Via Afrika Mathematics

Grade 10 Study Guide

M. Pillay, L.J. Schalekamp, G. du Toit, C.R. Smith, L. Bruce, M.L. Botsane, J. Bouman, N.S. Muthige, S.F. Carletti

Via Afrika

Our Teachers. Our Future.

Authors • Authors • Authors • Authors

Study Guide

Mathematics

Grade 10

Via Afrika Publishers

Our Teachers. Our Future.

Contents

Introduction to Mathematics

There once was a magnificent mathematical horse. You could teach it arithmetic, which it learned with no difficulty, and algebra was a breeze. It could even prove theorems in Euclidean geometry. But when it tried to learn analytic geometry, it would rear back on its hind legs, and make violent head motions in resistance.

The moral of this story is that you can't put Descartes before the horse. If you have a study routine that you are happy with and you are getting the grade you want from your mathematics class you might benefit from comparing your study habits to the tips presented here.

Mathematics is Not a Spectator Sport

In order to learn mathematics you must be actively involved in the doing and feeling of mathematics.

Work to Understand the Principles

- **LISTEN During Class.** No the study guide is not enough. In order to get something out of the class you need to listen while in class. Sometimes important ideas will not be written down, but instead be spoken by the teacher.
- **Learn the (proper) Notation**. Bad notation can jeopardise your results. Pay attention to them in the worked examples in this study guide,
- **Practise, Practise, Practise**. Practise as much as possible. The only way to really learn how to do problems is work through lots of them. The more you work, the better prepared you will be come exam time. There are extra practise opportunities in this study guide.
- **Persevere**. You might not instantly understand every topic covered in a mathematics class. There might be some topics that you will have to work at before you completely understand them. Think about these topics and work through problems from this study guide. You will often find that, after a little work, a topic that initially baffled you will suddenly make sense.
- **Have the Proper Attitude**. Always do the best that you can.

The AMA of Mathematics

- **ABILITY** is what you're capable of doing.
- **MOTIVATION** determines what you do.
- **ATTITUDE** determines how well you do it.

It is not pure intellectual power that counts, it's commitment. – Dana Scott

Chapter 1

Algebraic expressions

Overview

In this unit, we discuss real numbers, which are divided into rational and irrational numbers. Here, you will also learn about surds, and how to round off real numbers. You will also learn how to multiply integers, monomials and binomials by a polynomial. Finally, we discuss factorisation and how to work with algebraic fractions.

CHAPTER 1 Page 6 – Algebraic expressions

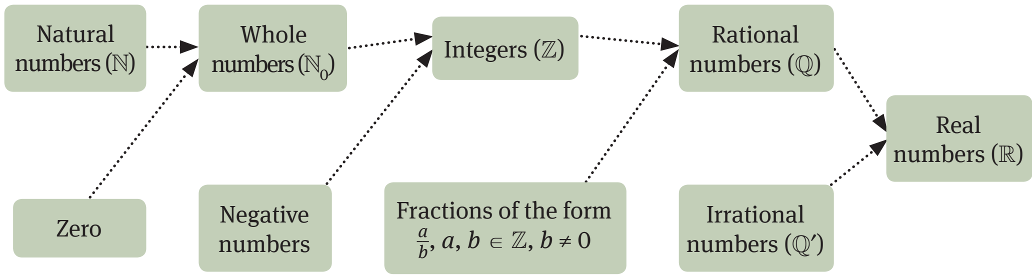
- **UNIT 1 Page 7** – The number system
  - Real numbers
  - Surds
  - Rounding off
- **UNIT 2 Page 8** – Multiplying algebraic expressions
  - Multiply integers and monomials by polynomials
  - The product of two binomials
  - Multiplying a binomial by a trinomial
  - The sum and difference of two cubes
- **UNIT 3 Page 10** – Factorisation
  - Common factors
  - Difference between two squares
  - Perfect squares
  - Trinomials of the form $x^2 + bx + c$
  - Trinomials of the form $ax^2 + bx + c$
  - Factorising by grouping
  - Factorising the sum and difference of cubes
- **UNIT 4 Page 13** – Algebraic fractions
  - Simplifying fractions
  - Products of algebraic fractions
  - Adding and subtracting algebraic fractions

Unit 1

The number system

1.1 Real numbers

- Real numbers are divided into rational and irrational numbers.



- We can write a rational number as a fraction, $\frac{a}{b}$, where a and $b \in$ Z, and $b \neq 0$.
- We cannot express an irrational number as a fraction.

1.2 Surds

- A surd is a root of an integer that we cannot express as a fraction.
- Surds are irrational numbers.
- Examples of surds are $\sqrt{3}$ and $\sqrt{2}$.

1.3 Rounding off

- If the number after the cut-off point is 4 or less, then we leave the number before it as it is.
- If the number is equal to 5 or more, then increase the value of the number before it by 1.

Unit 2

Multiplying algebraic expressions

2.1 Multiply integers and monomials by polynomials

- Each term inside the bracket is multiplied by the term in front of the bracket.

$$-\tfrac{1}{6}a^2b(6abc - 12ac + 18b) = (-\tfrac{1}{6}a^2b \times 6abc) - (-\tfrac{1}{6}a^2b \times 12ac) + (-\tfrac{1}{6}a^2b \times 18b)$$
$$= -a^3b^2c + 2a^3bc - 3a^2b^2$$

2.2 The product of two binomials

- Each term inside the first set of brackets is multiplied by each term inside the second set of brackets.

$$(a - 3b)(a + 7b) = (a \times a) + (a \times 7b) - (3b \times a) - (3b \times 7b)$$
$$= a^2 + 7ab - 3ab - 21b^2$$
$$= a^2 + 4ab - 21b^2$$

- When squaring a binomial:

$$(m + n)^2 = (m + n)(m + n)$$
$$= (m \times m) + (m \times n) + (n \times m) + (n \times n)$$
$$= m^2 + mn + nm + n^2$$
$$= m^2 + 2mn + n^2$$

A common error is to think that $(m + n)^2 = m^2 + n^2$

The square of any binomial produces the following three terms:

1 The square of the first term of the binomial: m^2

2 Twice the product of the two terms: $2mn$

3 The square of the second term: n^2

2.3 Multiplying a binomial by a trinomial

Multiply each term in the first set of brackets by each term in the second set of brackets. Then we simplify by collecting the like terms.

$$(8 - 3y)(12 - 2y + 8y^2 - 4y^3) = (8 \times 12) + (8 \times -2y) + (8 \times 8y^2) + (8 \times -4y^3) + (-3y \times 12)$$
$$+ (-3y \times -2y) + (-3y \times 8y^2) + (-3y \times -4y^3)$$
$$= 96 - 16y + 64y^2 - 32y^3 - 36y + 6y^2 - 24y^3 + 12y^4$$
$$= 96 - 52y + 70y^2 - 24y^3 + 12y^4$$

2.4 The sum and difference of two cubes

- The expression in the following form gives the **difference** of two cubes:

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a^3 + a^2b + \ \times ab \times {}^2 - a^2b - \ \times ab \times {}^2 - b^3 \\ &= a^3 - b^3\end{aligned}$$

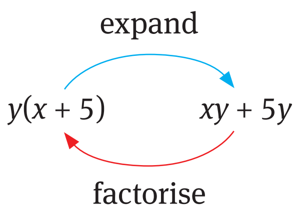
- The expression in this form gives the **sum** of two cubes:

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 \\ &= a^3 + b^3\end{aligned}$$

Unit 3

Factorisation

Factorisation is the opposite process to the one you learnt in the previous unit when multiplying out algebraic expressions.



3.1 Common factors

When finding a common factor, we find and take out a factor that can divide into each term in the expression. For example, in the expression $6h^3 + 18h^2$, $6h^2$ can divide into each term, so it becomes the common factor:

$6h^3 \div 6h^2 = h$
$18h^2 \div 6h^2 = +3$

Therefore:
$6h^3 + 18h^2 = 6h^2(h + 3)$

3.2 Difference between two squares

A difference of two squares occurs when we have two perfect squares separated by a minus sign. An expression in the form $a^2 - b^2$ has two factors, $(a - b)(a + b)$. For example:

$4a^2 - 81b^4 = (2a + 9b^2)(2a - 9b^2)$

$\sqrt{4a^2} = 2a$
$\sqrt{81b^4} = 9b^2$

3.3 Perfect squares

In Unit 2 you learnt that $(x + y)^2 = x^2 + 2xy + y^2$. Therefore, a trinomial will factorise into a perfect square if:

- the first and last terms are perfect squares (e.g. $x \times x = x^2$)
- the middle term is equal to 2 × first term × last term.

For example: $x^2 + 10x + 25 = (x + 5)(x + 5) = (x + 5)^2$

3.4 Trinomials of the form $x^2 + bx + c$

When factorising a trinomial:

- If the third term of the trinomial is positive, we use the sum of the factors of the third term to give us the middle term.
- If the third term of the trinomial is negative, we use the difference between the factors of the third term to give us the middle term.

Use the following rules to decide on the signs in the binomials:

- If the third term of the trinomial is positive, then the signs between the terms in the binomials are the same (both positive or negative).
- If the third term of the trinomial is negative, then the signs between the terms of the binomials are different.

For example: $x^2 + 3x + 2 = (x + 2)(x + 1)$

3.5 Trinomials of the form $ax^2 + bx + c$

In this case, the rules are the same as in the previous section, except that now we need to consider the factors of the coefficient of x^2 and the last term.

For example: $6x^2 + x - 15$

- Multiply the coefficient of the squared term (6) by the last term $(-15) = -90$.
- Look for the factors of -90 that will give $+1$ (the coefficient of the middle term) when added:
  $-9 \times 10 = -90$ and $-9 + 10 = 1$
- Rewrite $6x^2 + x - 15$ as $6x^2 + 10x - 9x - 15$.
- Group the terms: $(6x^2 + 10x) - (9x + 15)$
- Take out common factors from each set of brackets:
  $2x(3x + 5) - 3(3x + 5) = (3x + 5)(2x - 3)$

Remember to change the sign inside the brackets when you divide by -1.

Therefore:

$$\begin{aligned} 6x^2 + x - 15 &= 6x^2 + 10x - 9x - 15 \\ &= 2x(3x + 5) - 3(3x + 5) \\ &= (3x + 5)(2x - 3) \end{aligned}$$

3.6 Factorising by grouping

If an expression has four or more terms, but has no factor common to all of them, we can often group the terms, factorise each group, and then remove a common factor.

$x^3 - 3x^2 + 2x - 6 = (x^3 - 3x^2) + (2x - 6)$ (Group terms together using brackets)
$= x^2(x - 3) + 2(x - 3)$ (Factor out the common factors from each group)
$= (x - 3)(x^2 + 2)$

3.7 Factorising the sum and difference of cubes

- The **difference** between two cubes factorises as: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$
  - The first factor is: $(\sqrt[3]{\text{first term}}) - (\sqrt[3]{\text{second term}})$
  - The second factor is: (Square of the first term) (Opposite sign) (Product of the two terms) + (Square of the last term)
    For example: $y^3 - 64 = (y - 4)(y^2 + 4y + 16)$
- The **sum** of two cubes factorises as: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
  For example: $y^3 + 64 = (y + 4)(y^2 - 4y + 16)$

Unit 4

Algebraic fractions

4.1 Simplifying fractions

Special fractions:	Negative fractions:
$\frac{a}{1} = a$ $\frac{1}{a} = \frac{1}{a}$ $\frac{a}{a} = 1$ $\frac{0}{a} = 0$ *But*: $\frac{a}{0}$ is undefined	$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$ $\frac{-a}{-b} = \frac{a}{b}$ *But*: $-\frac{a}{b} \neq \frac{-a}{-b}$

To simplify a fraction:

1 Factorise the expressions in the numerator and denominator

2 Cancel the terms common to both.

For example: $\frac{(x^2 - 4)}{(x - 2)} = \frac{(x + 2)(x - 2)}{(x - 2)} = (x + 2)$

4.2 Products of algebraic fractions

Multiplying and dividing algebraic fractions works in the same way as ordinary fractions that contain numbers:

- Factorise each expression.
- Cancel any like terms.
- Multiply what is left.

$$\frac{(4n^2 - 9)}{(2n + 3)} \times \frac{(2n^2 - n - 3)}{(n + 1)} = \frac{(2n + 3)(2n - 3)}{(2n + 3)} \times \frac{(2n - 3)(n + 1)}{(n + 1)}$$

$$= \frac{(2n - 3)}{1} \times \frac{(2n - 3)}{1}$$

$$= (2n - 3)^2$$

Multiply each term in the first set of brackets by each term in the second set of brackets. Then we simplify by collecting the like terms.

4.3 Adding and subtracting algebraic fractions

- As always, factorise and simplify where possible.
- Find a common denominator.
- Express each fraction in terms of that denominator.
- Add and subtract the like terms.

$\frac{3p-q}{2p}+\frac{2p-3q}{4q}$ $=\frac{2q(3p-q)+p(2p-3q)}{4pq}$ $=\frac{6pq-2q^2+2p^2-3pq}{4pq}$ $=\frac{2p^2+3pq-2q^2}{4pq}$ $=\frac{(2p-q)(p+2q)}{4pq}$	$\frac{2x+5}{5}-\frac{x-2}{3}$ $=\frac{3(2x+5)-5(x-2)}{15}$ $=\frac{6x+15-5x+10}{15}$ $=\frac{x+25}{15}$

Questions

1 Multiplying expressions:

a Square the binomial $(3x - 4)$.

b Expand: $(2x - 5)(2x + 1)$

c Expand: $(2a + b)(4a^2 - 2ab + b^2)$

d Find the products and simplify: $(2a - 5b)(4a - 3b) - (a + 3b)(5a - 12b)$

e Find the product: $(k^2 + \frac{3}{4})(k - \frac{1}{2})$

2 Factorise the following expressions:

a $25x^2 + 30x + 9$

b $5x^2 - 7x - 6$

c $x^3 - \frac{1}{4}x$

d $(2t - 5)^3 - (2t - 5)^2$

e $3x^3 + x^2 - 3x - 1$

f $m^3 - m^2 - mn^2 + n^2$

g $(y + 3)(2y - 3) + (y - 1)(3 - 2y)$

h $x^2(2x - 1) - 2x(2x - 1) - 3(2x - 1)$

3 Algebraic fractions:

a Simplify $\frac{x^2 - x - 6}{x^2 - 4x + 3}$

b Simplify to the lowest terms: $\frac{4x^3 - 9x^2}{4x^3 + 6x^2}$

c Simplify as far as possible: $\frac{x^2 - 7x + 12}{4 - x}$

d Simplify as far as possible: $\frac{a(a + 1) + (a + 1)}{a^2 - 2a + 1} \times \frac{a^2 - 1}{a^2} \div \frac{a^2 + a}{a^2 - a}$

e Calculate and simplify: $\frac{1}{x - 2} + \frac{-4}{(x + 2)^2} - \frac{1}{x + 2}$

f Calculate and simplify: $\frac{p - 3}{p^2 - p - 12} + \frac{2}{3 + p} - \frac{3}{8 - 2p}$

Chapter 2

Exponents

Overview

As you know, we can describe numbers in terms of their factors. For example, $125 = 5 \times 5 \times 5$. We can also write this another way, in exponential form: $125 = 5^3$. Here, 5 is the base, and 3 is the exponent. An exponent is also sometimes called an index or a power. In this chapter, you learn about the laws of exponents, how to simplify exponents and how to solve equations that contain exponents.

Unit 1

Laws of exponents

1.1 What is an exponent?

In general, we define exponents as follows:

$$x^n = x \times x \times x \times x \times \ldots \times x,\ n \in \mathrm{N}$$

This definition states that x is multiplied by itself n times. Here, x is the base, and n is the exponent.

1.2 The laws of exponents

- $b^m \times b^n = b^{(m+n)}$ (The product rule)
- $b^m \div b^n = b^{(m-n)}$ where $m \geq n$ and $b \neq 0$ (The quotient rule)
- $a^0 = 1$, when $a \neq 0$
- $(b^m)^n = b^{mn}$ (The power rule)
- $(ab)^n = a^n b^n$
- $a^{-1} = \frac{1}{a}$, when $a \neq 0$

Note that, for these rules, the base stays the same.

Unit 2

Simplifying exponents

In this unit, we revise how to use the laws of exponents to simplify expressions.

2.1 The product rule: $b^m \times b^n = b^{(m+n)}$

The product rule tells you that, when **multiplying** powers that have the **same base**, you can **add** the exponents.

Example

1 $x^5 \times x^3 = x^{5+3} = x^8$

2 $2^3 \times 2^{-2} = 2^{3+(-2)} = 2^1 = 2$

3 $2 \times 2^{3x} \times 2^2 = 2^{1+3x+2} = 2^{3x+3}$

2.2 The quotient rule: $b^m \div b^n = b^{(m-n)}$

The quotient rule tells you that you can **divide** powers with the **same base** by **subtracting** the exponents.

Example

1 $\frac{x^5}{x} = x^{5-1} = x^4$

2 $\frac{2^3}{2^5} = 2^{3-5} = 2^{-2}$

3 $\frac{12a^6b^6}{-4a \times^{-2} b^3} = -3a^{6-(-2)}b^{6-3} = -3a^8b^3$

An exponent is also called a power.

2.3 Rules for 0 and 1

Rules

$x^1 = x$	$1^n = 1$	$1^0 = 1$
$12^1 = 12$	$1^4 = 1 \times 1 \times 1 \times 1 = 1$	$x^0 = 1, x \neq 0$

Anything to the power zero is equal to 1, except 0^0, which is undefined.

Example

1 $16a^0, a \neq 0$

2 $16 \times 1 = 16$

3 $(2 + x)^0 = 1$

4 $(-3)^0 - 3^0 = 1 - 1 = 0$ $[(-3)^0 = 1 \text{ and } 3^0 = 1]$

5 $\frac{3a^0}{2} + \left(\frac{b}{3a}\right)^0 = \frac{3}{2} + 1 = \frac{5}{2}$

Everything inside a bracket to the power of zero is always one.

2.4 Power rule: $(b^m)^n = b^{mn}$

The power rule tells us that to raise a power to a power, just **multiply the exponents**.

Example

1 $5(a^2)^3 = 5^{2 \times 3} = 5^6$

2 $(27)^{\frac{2}{3}} = (3^3)^{\frac{2}{3}} = 3^2 = 9$

3 $(a^{-3}.b^{\frac{1}{3}}.c)^3 = a^{-3 \times 3}.b^{\frac{1}{3}} \times 3.c^{1 \times 3} = a^{-9}.b.c^3 = \frac{bc^3}{a^9}$

Always express answers with positive exponents.

2.5 All inside brackets raised to a power: $(ab)^n = a^n b^n$ and $(\frac{a}{b})^n = \frac{a^n}{b^n}$

Example

1 $(5a^2)^3 = 5^3 a^{2 \times 3} = 125a^6$

2 $(2x^2y)^5 = 2^5 x^{5 \times 2} y^{1 \times 5} = 32x^{10}y^5$

3 $\left(\frac{3a}{b^2}\right)^x = \frac{3^x a^x}{b^{2x}}$

4 $2^x.3^x = (2.3)^x = 6^x$

5 $(2a^6b)^3(3a^2b^3)^2 = (2^3a^{6(3)}b^3)(3^2a^{2(2)}b^{3(2)}) = (8a^{18}b^3)(9a^4b^6)$
$= 72a^{18+4}b^{3+6} = 72a^{22}b^9$

Switch the denominator and numerator and multiply.

2.6 Negative power rule: $a^{-n} = \frac{1}{a^{-n}}$, provided that $a \neq 0$

Any non-zero number raised to a negative power equals its reciprocal raised to the opposite positive power.

Example

1 $\frac{2^2}{3} = \frac{1}{2^2 - 3} = \frac{1}{12}$

2 $\frac{b}{a^5} = \frac{b}{\frac{1}{a^5}} = b \times \frac{a^5}{1} = a^5b$

3 $\frac{a^2x^2}{a^3x^1} = \frac{\frac{1}{a^2}x^2}{a^3\frac{1}{x}} = \frac{\frac{x^2}{a^2}}{\frac{a^3}{x}} = \frac{x^2}{a^2} + \frac{x}{a^3} = \frac{x^3}{a^5}$

4 $\frac{3x^3}{(3x)^2} = \frac{3}{9x^2.x^3} = \frac{1}{3x^5}$ $\quad \frac{4a^2b^3}{3ab^1} = \frac{4b^4}{3a^3}$

Unit 3

Solving equations that contain exponents

The key to solving equations that contain exponents is to ensure that the bases on both sides of the equation are the same. This enables us to compare the exponents, and so solve for the variable. The rule is:
If $a^x = a^n$, then $x = n$

Example

1 $3^x = 81 = 3^4$ [Rewrite 81 in exponential form: $81 = 3^4$]
Therefore, $x = 4$

2 $7^{x-3} = 49$ [Rewrite 49 in exponential form: $49 = 7^2$]
$7^{x-3} = 7^2$
$x - 3 = 2$ [Exponents are equal if bases are equal]
$x = 5$

3 $3.9^{x-1} = 27^{-x}$ [Make the bases the same by applying the laws of exponents]
$3.3^{2(x-1)} = 3^{3(-x)}$
$3^{1+2x-2} = 3^{-3x}$
$3^{2x-1} = 3^{-3x}$
$2x - 1 = -3x$
$2x + 3x = 1$
$5x = 1$
$x = \frac{1}{5}$

4 $2^x + 1 = 9$ [Move terms without x to the right side of the equation]
$2^x = 9 - 1$
$= 8$
$2^x = 2^3$ [$8 = 2^3$]
Therefore, $x = 3$ according to the principle of same bases.

3.1 Solving exponential equations using factorisation

Before we remove a common factor from an expression, we sometimes need to apply the laws of exponents in reverse.

Example

1 $2^{x+1} = 2^x = 24$ [Use the reverse of the product rule: $2^{x+1} = 2^x \times 2^1$]
$2^x.2 + 2^x = 24$ [Take out a common factor of $2x$]
$2^x(2 + 1) = 24$
$3.2^x = 24$
$2^x = \frac{24}{3}$

$2^x = 8$

$2^x = 2^3$

$x = 3$

2 $3^{x+1} = 3^{x-1} = 90$

$3^x(3^1 + 3^{-1}) = 90$ [Take out a common factor of $3x$]

$3^x(3 + \frac{1}{3}) = 90$

$3x\left(\frac{10}{3}\right) = 90$

$3x = 90 \quad \frac{3}{10}$

$3x = 27$ [The bases are the same, so exponents are equal]

$3x = 3^3$

$x = 3$

3 $2^{x+1} = 2^x + 4$ [Move terms containing x to the left side of the equation]

$2^{x+1} - 2^x = 4$

$2x(2 - 1) = 4$ [Take out a common factor of $2x$]

$2^x = 4(4 \div 1 = 4)$

$2^x = 2^2$

Therefore, $x = 2$

3.2 Exponents containing fractions

Fractions often make expressions look more complicated, but we use all the same rules and laws of exponents.

Example

1 $4^{\frac{5}{2}} = (2^2)^{\frac{5}{2}} = 2^5 = 32$

2 $(3^3x^{-12}) = 3^2x^{-8} = \frac{9}{x^8}$ [Remember to express exponents as positive numbers]

3 $\frac{1}{27}^{\frac{1}{3}} = (3^3)^{\frac{1}{3}} = 3^1 = 3$

4 $(27x^{12})^{\frac{2}{3}} = (3^3)^2.(x^{12})^2 = 3^2x^8 = \frac{9}{x^8}$

5 $\frac{2^y a^{2y}}{b^{\frac{y}{2}}} = \frac{2^{y\frac{2}{y}}.a^{2y\frac{2}{y}}}{b^{\frac{y}{2}-\frac{2}{y}}} = \frac{2^2a^4}{b^1} = \frac{4a^4}{b}$

3.3 Three common mistakes to avoid when working with exponents

3.3.1 Brackets

The exponent next to a number only applies to that number. However, if the numbers are inside brackets, and the exponent lies outside the brackets, then the exponent applies to everything inside the brackets. For example:

$-3^2 \neq (-3)^2 \qquad -3 \times 3 \neq (-3)(-3)$

3.3.2 The product rule

1. The product rule only applies when the bases are the same. For example: $2^3 \times 3^2 \neq 6^5$
2. The product rule only applies when multiplying, not when adding. For example: $2^3 + 2^5 \neq 2^8$

3.3 Working backwards to go forwards

Remember, we need the bases to be the same before we can apply the laws of exponents. Therefore, first write all bases as a product of their **prime factors**, and then apply the laws of the exponents to those bases that are the same.

$27 = 3^3$ $81 = 3^4$ $32 = 2^5$ $36 = 2^2 \times 3^3$ $40 = 2^3 \times 5^1$

Example

$9 = 3^2$
$12 = 3 \times 4 = 3.2^2$
$6 = 2 \times 3$
$4 = 2^2$

Therefore:

$$\frac{9^n \times 12^{n+1}}{4 \times 6^n} = \frac{(3^2)^n \times (3.2^2)^{n+1}}{2^2 \times (2.3)^n}$$
$$= \frac{3^{2n} \times 3^{n+1} \times 2^{2n+2}}{2^2 \times 2^n \times 3n}$$
$$= 3^{2n+n+1-n} \times 2^{2n+2-2-n}$$
$$= 3^{2n+1} \times 2n$$

Questions

1 Simplify the following expressions by applying the laws of exponents. All indices must be positive, and all variables represent positive rational numbers.

a $(2a^n - 2)^2$

b $5x^0 + 8^{\frac{-2}{3}} - \left(\frac{1}{2}\right)^{-2}.1x$

c $\frac{2a^{-2}3^a}{6^a}$

d $\frac{12x^2.x^4 - (-3x^3)^2}{3x^3}$

e $\frac{27^{x+1}.9^{x-1}}{3^{x-1}.81^{x+1}}$

f $\frac{2^{-3}x}{3^{-1}x^{-2}}$

g $\frac{4^{a-1}.8^{a+1}}{2^{a-2}.6a+1}$

h $\frac{1}{8^{\frac{-2}{3}}} - 3a^0 + 27^{\frac{1}{3}} - 1^{\frac{2}{3}}$

i $\frac{x^{2a-b}}{x^{b-2a}} \div \frac{x^{4a}}{x^{2b}}$

2 a Calculate b if $b = \frac{5^{n-3}}{5^{n+1}}$

b If $5x = f$, express 2.25^{x+1} in terms of f.

c If $3^m = 5$, find the value of

i 3^{m+2}

ii 9^{-2m}

d Remove the brackets and simplify: $(3^{2x} - 3)(3^{2x} + 3)$

3 Solve the following equations:

a $2^x + 1 = 2$

b $9^{x-2} = 27^{1-2x}$

c $5^{2x+1} = 0{,}04$

d $5^{x+1} + 5^x - 150 = 0$

4 An astronomer studying a region of space needs to determine the volume of a cubic region. The edges of the region measure 3×108 miles long. Find the volume.

Chapter 3

Number patterns

Overview

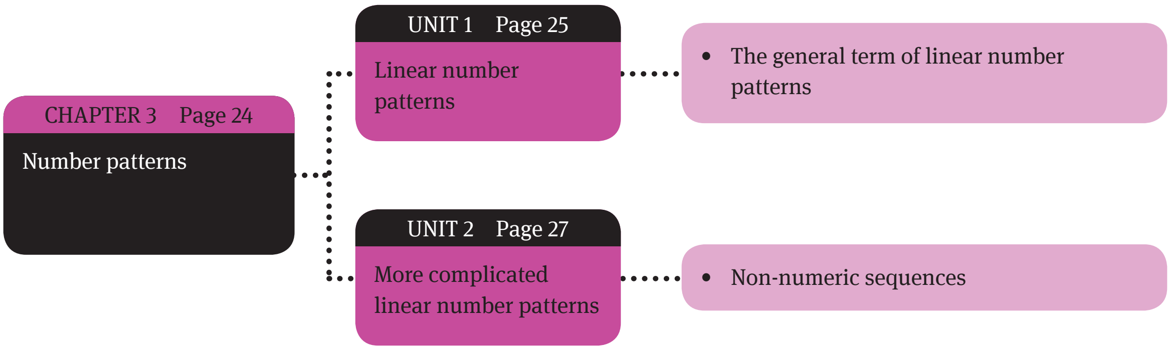
A number pattern is a list of numbers. The numbers are in a definite order, and follow a fixed rule, or pattern. Examples of number patterns are:

- Even numbers 2; 4; 6; 8; 10; 12; ...
- Odd numbers 1; 3; 5; 7; 9; 11; ...
- Square numbers 1; 4; 9; 16; 25; 36; 49; ...
- Cube numbers 1; 8; 27; 64; 125; 216; ...
- Fibonacci numbers 1; 1; 2; 3; 5; 8; 13; 21; 34; ...

In a number pattern:

- the numbers in the sequence are called terms
- each term has a position in the sequence.

A linear number pattern is a special kind of number pattern, in which the difference between the terms is always a fixed value.



Unit 1

Linear number patterns

1.1 The general term of linear number patterns

A linear number pattern has a constant difference between consecutive terms.

For example, consider the pattern: 2; 5; 8; 11; ... We obtain each term of the pattern by adding 3 to the previous term. Therefore, the difference between the second and first term is 3. The difference between the third and the second term is also 3, and so on.

We represent the first term in the pattern by the letter a. We represent the constant difference between terms as d. Therefore, in our previous example:

$a = 2$ (first term)
$d = 3$ (constant difference)

We define the first term (a) as T_1, the second term as T_2, the third term as T_3, and so on. In our example, It is relatively easy to find the sixth term in the pattern (17), but it is not as easy to find, for example the 1 000th term. To do so, we need to develop a rule, or formula.

The general formula to calculate the value of a term in a linear number pattern is:

$$T_n = a + (n - 1) \times d$$

Therefore, the 1 000th term is: $T_{1\,000} = 2 + (999) \times 3 = 2\,999$

Example

The general term (T_n) of a sequence is given by $T_n = 6n + 2$. Find the first four terms.	$T_1 = 6(1) + 2 = 8$ Replace n with "1" for Term 1 $T_2 = 6(2) + 2 = 14$ Replace n with "2" for Term 2 $T_3 = 6(3) + 2 = 20$ $T_4 = 6(4) + 2 = 26$ $\therefore$ sequence is 8; 14; 20; 26; ...
Given the sequence 3; 7; 11; 15 ... Determine the formula for T_n.	3 ; 7 ; 11 ; 15 ... +4 +4 +4 Constant difference $T_n = an + d$ a = constant difference Therefore: $T_n = 4n + d$ Substitute $n = 1$: $T_1 = 4(1) + d = 3$ $\therefore b = -1$ Therefore, $T_n = 4n - 1$

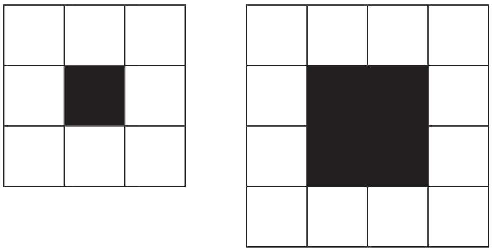
Given the sequence 10; 2; −6; −14; … Find T_n.	10 ; 2 ; −6 ; −14 −8 −8 −8 Constant difference $T_n = an + d$ a = constant difference Therefore: $T_n = -8n + d$ Substitute $n = 1$: $T_1 = -8(1) + d = 10$ $b = 18$ Therefore, $T_n = -8n + 18$

Unit 2

More complicated linear number patterns

2.1 Non-numeric sequences

Matches are used to build **squares** around a **central area (in black)**, as shown below. Complete the table that shows how wide the squares are, and how many matches are used in each square.



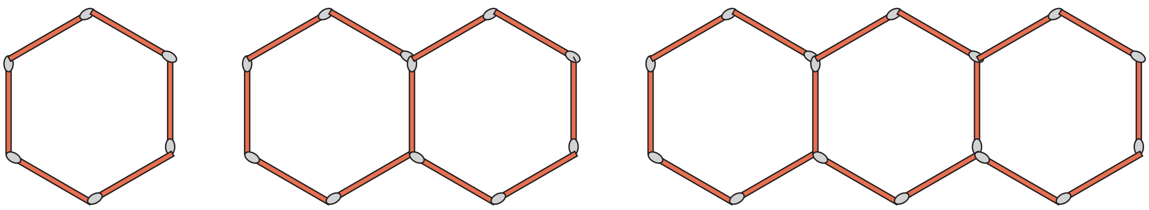
Number of matches wide	3	4	5	7	156	n
Number of matches used	24	36	48	?	?	?

There is a constant difference of 12 between matches used. Therefore:

$24 = 12(1) + d$
$d = 12$
$T_n = 12n + 12 = 12(n + 1)$

Term 7 = 12(7) + 12 = 96 matches (if the square is 27 matches wide)
Term 156 = 12 (157) = 1 884 matches (if the square is 156 matches wide)

Rapunzel created a chain by forming hexagons with matches. She predicts that the number of matchsticks needed to form a chain of 20 hexagons will be 121.



To test this prediction, we need a rule:
H1 = 6 matches
H2 = 6 + 5 = 11 matches
H3 = 6 + 5 + 5 = 16 matches

Note that 6 is the starting value and then we 5 each time, one less than the number of hexagons. Therefore:

$$H_n = 6 + (n - 1) \times 5 = 5n + 1$$

Apply the rule for 20 hexagons: $5(20) + 1 = 101$
Therefore, the prediction is false.

Questions

1 Complete the following patterns by writing down the next two terms.

a 12 ; 19 ; 26 ; ...

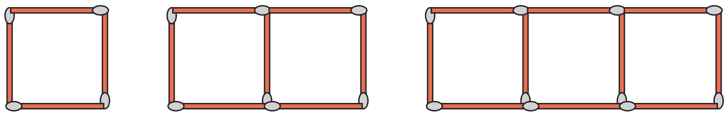
b $\frac{2}{3}$; $\frac{3}{5}$; $\frac{4}{7}$; ...

2 Determine the *n*th term and 16th term of the following patterns:

a 25; 22; 19; 16;...

b 1; 2; 4; 8; 16;...

3 Chains of squares can be built with matchsticks as shown below.



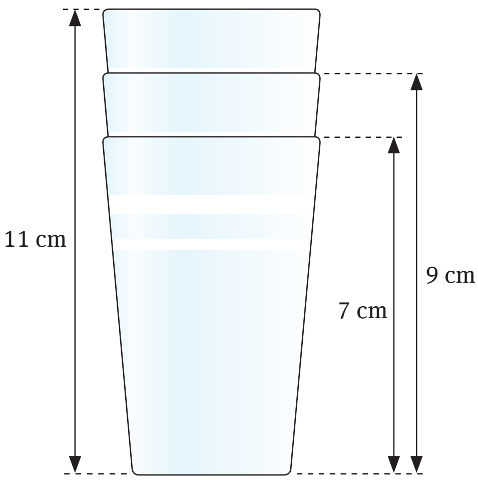
a How many matches are used to create a chain of 4 squares?

b How many matches are used to create a chain of 5 squares?

c What is the general formula for calculating how many matches there will be in a chain of squares?

d Now determine how many matches will be needed to build a chain of 100 squares.

4 You are stacking polystyrene cups, which fit into each other as in the diagram below. The first cup is 7 cm high. Two cups stacked together are 9 cm high, and three cups stacked together are 11 cm high.



a How high would a stack of four cups be?

b Write down the sequence of numbers in stacks with 1, 2, 3, 4, 5 and 6 cups.

c Find a general formula for the height of a stack of *n* cups.

d How many cups will stack on a shelf if the height between shelves is 22 cm?

e Can you build a stack of cups that has a height of 81 cm? If so, how many cups do you need?

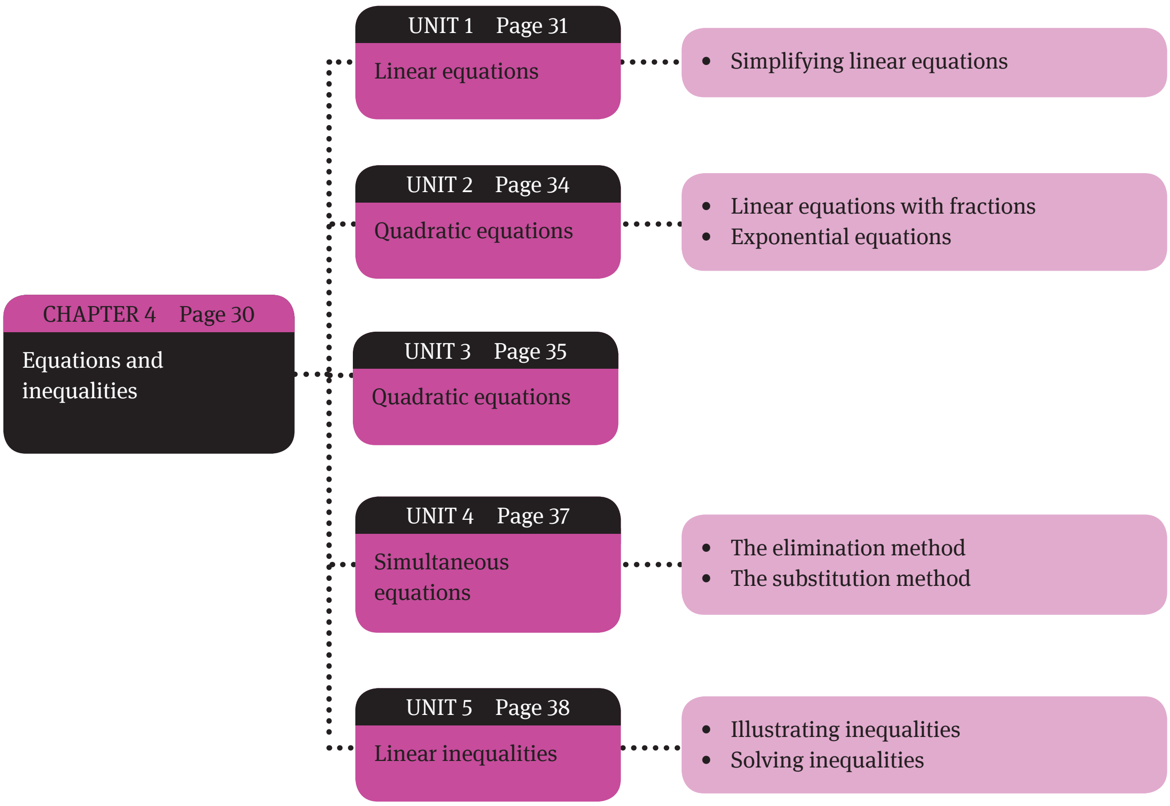
Chapter 4

Equations and inequalities

Overview

In this chapter, we investigate equations and inequalities. We begin with linear equations before moving on to more compicated equations. An inequality is similar to an equation, except that it contains an inequality sign rather than an equals sign. The rules are very similar, but there are also a few important differences. So here you will learn:

- how to solve linear equations
- how to solve quadratic equations
- how to solve simultaneous equations
- how to work with linear inequalities, including how to represent them on a number line.



Unit 1

Linear equations

1.1 Simplifying linear equations

Use the following steps to solve a linear equation:

Steps to follow	Example
Simplify by removing brackets and gathering like terms.	$3(5x - 2) = 2(4x + 11)$ $15x - 6 = 8x + 22$
Add and subtract terms on both sides until all variables are on the left of the equals sign and the constant terms are on the right.	$15x - 8x = 22 + 6$ $7x = 28$
Divide both sides by the coefficient of the variable (x).	$\frac{7x}{7} = \frac{28}{7}$ $\therefore x = 4$
Check the solution by substituting the value back into the original equation.	LHS: $3(5 \times 4 - 2) = 3(20 - 2) = 3(18) = 54$ RHS: $2(4 \times 4 + 11) = 2(16 + 11) = 2(27) = 54$ Since LHS = RHS, the solution is correct.

Special cases: Identities and false statements

The following statement is true for **all** values of x, and so we call it an **identity**.

$$2(3x + 4) = 6x + 8$$
$$\therefore 6x + 8 = 6x + 8$$
$$\therefore 6x - 6x = 8 - 8$$
$$\therefore 0 = 0$$
$$\therefore x \in \mathbb{R}$$

The following statement can never be true, so we call it a **false statement**.

$$2(3x + 4) = 6x - 8$$
$$\therefore 6x + 8 = 6x - 8$$
$$\therefore 6x - 6x = -8 - 8$$
$$\therefore 0 = -16$$
$$\therefore x \notin \mathbb{R}$$

1.2 Linear equations with fractions

When working with fractions, always remember that you cannot divide by 0.

Steps to follow	Example
Write down any restrictions (that is, values for x that will make the denominator 0): $x \neq \pm 3$	$\frac{3}{x^2 - 9} + \frac{2}{x + 3} = \frac{1}{x - 3}$
Multiply both sides of the equation by the LCM of all the denominators: $(x + 3)(x - 3)$	$\frac{3}{x^2 - 9} + \frac{2}{x + 3} = \frac{1}{x - 3}$ $\frac{3}{(x + 3)(x - 3)} + \frac{2}{x + 3} = \frac{1}{x - 3}$ $\therefore 3 + 2(x - 3) = x + 3$

Steps to follow	Example	
Simplify the equation (remove the brackets and collect the like terms together)	$\therefore\ 3 + 2x - 6 = x + 3$ $\therefore\ 2x - x = 3 - 3 + 6$ $\therefore\ x = 6$	
Check that the answer is allowed (by referring back to any restrictions)	Yes, $x \neq \pm 3$	
Check the solution by substituting it into the original equation.	LHS: $= \frac{3}{x^2 - 9} + \frac{2}{x + 3}$ $= \frac{3}{6^2 - 9} + \frac{2}{6 + 3}$ $= \frac{3}{36 - 9} + \frac{2}{9}$ $= \frac{3}{27} + \frac{2}{9}$ $= \frac{1}{3}$	RHS: $= \frac{1}{6 - 3}$ $= \frac{1}{3}$
	LHS = RHS, therefore the solution is correct.	

Example

1 $\frac{x + 2}{x^2 - 2x} = \frac{1}{x - 2}$ [Restriction: $x \neq 0$; $x \neq 2$]

$(x - 2)(x + 2) = 1(x^2 - 2x)$

$x^2 - 4 = x^2 - 2x$

$2x = 4$

$x = 2$

However, $x \neq 2$, therefore there is no solution.

2 $\frac{3}{a - 1} - \frac{2}{a + 1} = 0$ [Restriction: $a \neq 1$; $a \neq -1$]

$3(a + 1) - 2(a - 1) = 0$

$3a + 3 - 2a + 2 = 0$

$a = -5$

Check solution: $\frac{3}{-5 - 1} - \frac{2}{-5 + 1} = -\frac{1}{2} + \frac{1}{2} = 0$

3 $\frac{x - 3}{2x} + \frac{x - 2}{3x} = 1$ [Restriction: $x \neq 0$]

$3(x - 3) + 2(x - 2) = 6x$

$3x - 9 + 2x - 4 = 6x$

$-x = 13$

$x = -13$

Check solution: $\frac{-13 - 3}{-26} + \frac{-13 - 2}{-39} = \frac{16}{26} + \frac{15}{39} = \frac{8}{13} + \frac{5}{13} = 1$

1.3 Exponential equations

You learnt how to solve exponential equations in Chapter 2. In this section, we will show you how to solve an equation when you are not able to make the bases the same on both sides of the equation. In this case, you need to use the **trial-and-error method**, as shown in the following example.

Example	Help table
$2^{x-1} = 13$ so, $x - 1 = 3{,}7$ $x = 2{,}7$	see table below

Help table

x	$x - 1$	Accuracy
5	16	Too big
4	8	Too small
3,5	11,313 …	Too small
3,6	12,125 …	Still too small
3,7	12,996 …	Very close
3,71	18,086	Too big

Unit 2

Quadratic equations

A quadratic equation has square term in it, for example, $x^2 + x - 12 = 0$. We say that the equation is of degree 2, or is a second-degree polynomial.

When you solve a quadratic equation, you may have 0, 1 or 2 solutions.

Steps to follow	Example
Remove any brackets and take all terms to the left of the equals sign so that there is only a 0 on the right. The equation will now be in the form: $ax^2 + bx + c = 0$	$2(2a^2 - 1) = 7a$ $4a^2 - 2 = 7a$ $4a^2 - 7a - 2 = 0$
Factorise the equation.	$(4a + 1)(a - 2) = 0$
We find the solution to the equation by letting each factor equal 0. Why? If $ab = 0$, then this statement will be true if $a = 0$ or if $b = 0$.	$(4a + 1) = 0$ or $(a - 2) = 0$ $\therefore a = -\frac{1}{4}$ or $a = 2$

Note: *Do not fall in the trap of trying to solve an equation without first removing the brackets and making the right-hand side of the equation equal to 0.*

Example

Solve for x: $x(x + 3) = 10$
Here, the solution is **not** $x = 10$ and $x + 3 = 10$.

The correct solution is:
$x^2 + 3x - 10 = 0$
$(x - 2)(x + 5) = 0$
$x = 2$ or $x = -5$

Unit 3

Literal equations

A literal equation is usually a formula, such as $E = mc^2$. Since there is more than one variable, we cannot solve a literal equation in the same way as an equation containing only one variable and numbers. What you are expected to be able to do is to change the subject of the formula. All this means is that you need to change the formula so that one variable appears on the left-hand side of the equation, and all the other variables and numbers are on the right-hand side of the equation.

Example

Steps to follow	Example
If needed, remove fractions by multiplying by the LCM (*abc*) of the denominators.	Solve for *x*: $\frac{x}{a} + \frac{1}{c} = \frac{x}{b}$ $bcx + ab = acx$
Write all the terms with the new subject on one side of the equals sign and all the other terms on the other side.	$bcx - acx = ab$
Factorise with the new subject (*x*) as a common factor.	$x(bc - ac) = ab$
Divide both sides by the coefficient (*bc* – *ac*) of the new subject.	$x = \frac{ab}{bc - ac}$

Let's work through a few more examples.

Steps to follow	Example
	Make *b* the subject of the formula: $3x = \sqrt{2b - y}$
If there is a square root (or cube root), square (or cube) both sides.	$(3x)^2 = (\sqrt{2b - y})^2$ $9x^2 = 2b - y$
Write all the terms with the new subject on one side of the equals sign, and all the other terms on the other side.	$2b = 9x^2 + y$
Divide both sides by the coefficient of the new subject.	$b = \frac{9x^2 + y}{2}$

Steps to follow	Example
	Make *r* the subject of the formula: $V = \frac{4}{3}\pi(R^3 - r^3)$
If needed remove fractions by multiplying by the LCM of the denominators.	$3V = 4\pi(R^3 - r^3)$
Multiply out the brackets to remove them.	$3V = 4\pi R^3 - 4\pi r^3$
Write all the terms with the new subject on one side of the equals sign, and all the other terms on the other side.	$4\pi r^3 = 4\pi R^3 - 3V$
Divide both sides by the coefficient of the new subject.	$r^3 = \frac{4\pi R^3 - 3V}{4\pi}$
Find the required root (cubed root) again, if necessary.	$r = \sqrt[3]{\frac{4\pi R^3 - 3V}{4\pi}}$

Steps to follow	Example
	Make h the subject of this formula: $T = 2\pi\sqrt{\frac{h}{g}}$
If there is square root (or cube root), square (or cube) both sides after placing the root sign on one side by itself.	$T = 2\pi\sqrt{\frac{h}{g}}$ $\frac{T}{2\pi} = \sqrt{\frac{h}{g}}$ $\left(\frac{T}{2\pi}\right)^2 = \left(\sqrt{\frac{h}{g}}\right)^2$ $\frac{T^2}{4\pi^2} = \frac{h}{g}$
Remove any fractions by multiplying both sides of the equation by the LCM of the denominators.	$4\pi^2 h = T^2 g$
Write all the terms with the new subject on one side of the equal sign and all the other terms on the other side. Divide both sides by the coefficient of the new subject.	$h = \frac{T^2 g}{4\pi^2}$

Unit 4

Simultaneous equations

Solving two equations simultaneously means finding one set of values for the variables that satisfy both equations. There are two ways to solve equations simultaneously (at the same time), namely the elimination method and the substitution method.

4.1 The elimination method

Steps to follow	Example
Rearrange the equations so that the coefficients of one of the variables are the same in both equations. **Tip: Number the equations** (The coefficient of y in both cases is now 4)	Solve the pair of equations simultaneously: $5x - 2y = 6$ .... (1) $3x + 4y = 14$ .... (2) Multiply equation 1 by 2: $10x - 4y = 12$ .... (3) [(1) × 2]
Add or subtract the equations to eliminate one of the variables.	Add (3) and (2): $13x = 26$
Solve for the unknown variable (x)	$x = 2$
Substitute this solution (x) back into either of the original equations to solve for the other variable (y).	Use $5x - 2y = 6$ .... (1) $5(2) - 2y = 6$ $-2y = 6 - 10$ $y = 2$ Therefore, $x = 2$ and $y = 2$.

4.2 The substitution method

Steps to follow	Example
	Solve the pair of equations simultaneously: $3x - y = 10$ .... (1) $3x - 4y = -8$ .... (2)
Rearrange at least one of the equations in the form $y = ...$ **Tip: Number the equations**	Use equation (1): $3x - y = 10$ $3x - 10 = y$ .... (3)
Substitute the value of y into the other equation.	Substitute (3) into (2): $3x - 4(3x - 10) = -8$
Solve for the unknown variable (x).	$3x - 12x + 40 = -8$ $3x - 12x = -8 - 40$ $9x = -48$ $x = \frac{48}{9}$ $x = 5\frac{1}{3}$
Substitute this solution (x) back into either of the original equations to solve for the other variable (y).	Use $3x - y = 10$: $3\left(\frac{16}{3}\right) - y = 10$ $16 - 10 = y$ $y = 6$ Therefore, $x = 5\frac{1}{3}$ and $y = 6$.

Unit 5

Linear inequalities

5.1 Illustrating inequalities

Linear inequalities have a range of solutions. As we will see later, we can represent the solution to an inequality graphically on a number line.

Example

Solve for x: $x + 3 < 2$

Even though we have an inequality rather than an equals sign, we use the same principles to solve the equation.

$x + 3 - 3 < 2 - 3$
$x < -1$

5.2 Solving inequalities

Note: *When multiplying or dividing an inequality by a negative number, you must reverse the direction of the inequality sign.*

Example

1 Solve for x: $\frac{x-4}{2} - \frac{5x+1}{3} \geq 1$
$3(x - 4) - 2(5x + 1) \geq 6$
$3x - 12 - 10x - 2 \geq 6$
$-7x \geq 6 + 12 + 2$
$-7x \geq 20$ [Divide by −7: reverse the inequality]
$x \leq \frac{20}{7}$

2 Solve for x: $3(x + 4) < 5x + 9$
$3x + 12 < 5x + 9$
$-2x < -3$ [Divide by −2: reverse the inequality]
$x > \frac{3}{2}$

As we mentioned earlier, we can show the solution of a linear inequality on a number line.

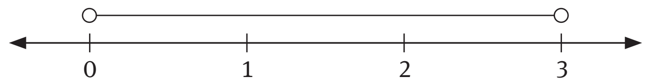
Example

Solve for n and indicate the solution on a number line: $-1 < 2 - n < 2$

$-1 - 2 < -n < 2 - 2$

$+3 > n > 0$

$0 < n < 3$



When the inequality is < or >, we show this as an open circle on the number line. When the inequality is ≥ or ≤, then we show this as a closed circle on the number line.

Questions

1 Solve the following equations and inequalities:

a $\frac{5a}{3} - 2 = \frac{a}{4} + 15$

b $\frac{m}{4} + 15 < \frac{5m}{3} - 2$

c $(2b + 1)(b + 8) = 27$

d $2 - \frac{1}{d-1} = \frac{3}{d+1}$

e $b^2 - 7b + 15 = 3$

f $\frac{5y-2}{3} + \frac{3y-1}{2} - \frac{y+1}{2} < -\frac{1}{3}$

g $\frac{3}{9h^2 - 16} - \frac{5}{3h^2 + 5h - 12} = \frac{2}{h(3h+4) + 3(4+3h)}$

2 Solve simultaneously: $2a = 24 + 7b$ and $3a + 5b = 5$

3 The formula for finding the volume, V, of material used to make a hollow sphere is given by $V = \frac{4}{3}\pi(R^3 - r^3)$, where R is the outer radius and r is the inner radius. Make R the subject of the formula, giving the expression in its simplest form.

4 Solve for in the following equations:

a $x^2 + 3(2x - 5) = x^2 - 2(7 + 3x)$

b $\frac{y - 3x}{x + z} = 2$

c $\frac{x+5}{6} + 1 = \frac{x-5-x}{3}$

d $\frac{3}{x^2 - 9} + \frac{2}{x+3} = \frac{1}{x-3}$

e $x - \frac{x-2}{3} \geq \frac{3}{2} + \frac{7x}{8}$

5. Solve for a and b:

$7(a + 2) + 3(b - 5) = 34$

and $3(a + 2) - 2(b - 5) = 8$

Chapter 5

Trigonometry

Overview

Here, we will introduce you to trigonometry. Trigonometry is the field of mathematics in which we study the relationship between the sides and angles of triangles. The word trigonometry is derived from the Greek words *trigonon*, which means triangle, and *metron*, which means measurement. In ancient times, mathematicians, astronomers and surveyors in Egypt, Babylon, India and China used trigonometry for navigation, surveying and astronomy. For example, they were able to calculate:

- the height of mountains
- the distance and direction across the sea (navigation)
- the dimensions of large areas of land for construction
- astronomical distances, for example, between the Earth, the moon and the sun.

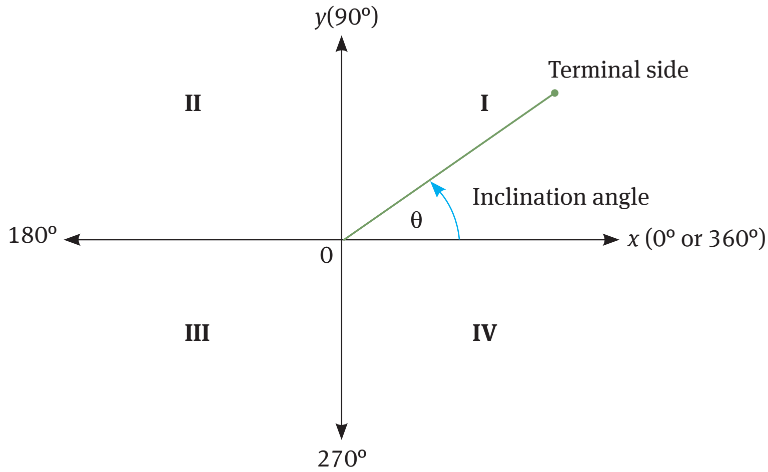
You will also learn how to sketch the graphs of each of the trigonometric ratios.

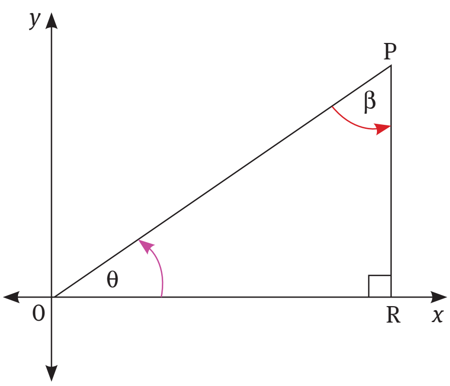
Unit 1

Right-angled triangles

1.1 Trigonometric ratios in right-angled triangles

Trigonometry is the measurement of triangles, specifically **right-angled triangles**. When we use the Cartesian plane in trigonometry, we call the positive x-axis the 0° line. If we move anti-clockwise by 360°, we have completed a full circle. The line that forms any angle (θ) in this way is known as the **terminal side** and the angle (θ) is known as the **inclination angle**.



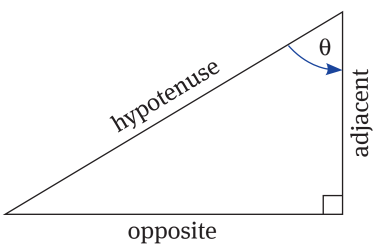


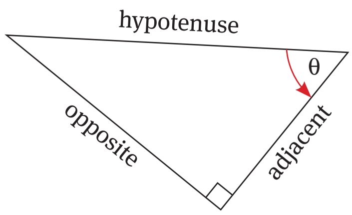
In a right-angled triangle, each side has a specific name:

- The *hypotenuse* (OP) is always opposite the 90° angle.
- The opposite side is opposite the angle. In this example, PR is opposite θ and OR is opposite β.
- The adjacent side forms the angle together with the hypotenuse. Therefore OR is adjacent to θ, and PR is adjacent to β.

Adjacent is always next to the angle.

Opposite is opposite the angle.





Unit 2

Definitions of the trigonometric ratios

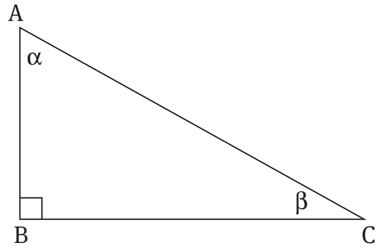
2.1 Definitions

In Trigonometry, the ratios between the sides are given certain names. In ΔABC with ∠BAC = α, we ca define the ratios as:

$$\sin\alpha = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\alpha = \frac{\text{opposite}}{\text{adjacent}}$$



2.2 Reciprocals of the three trigonometric ratios

The **reciprocal ratios** are the inverse versions of each of the main ratios. The inverse of sin θ is cosec θ, the inverse of cos θ is sec θ, and the inverse of tan θ is cot θ. The reciprocal ratios are defined as follows:

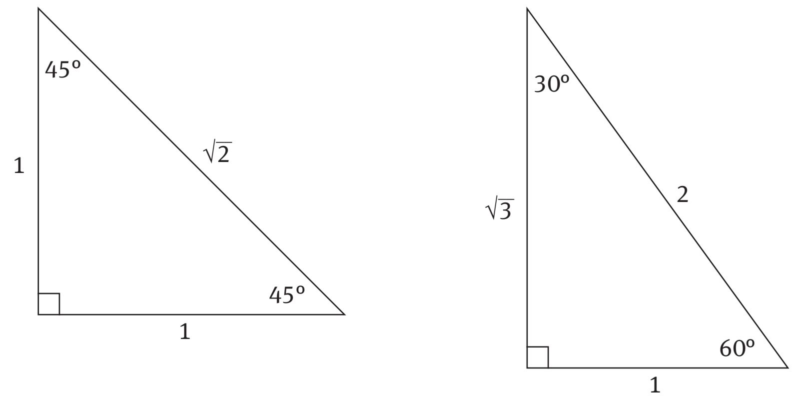
Trigonometric ratio	Reciprocal ratio	Conclusion
$\sin\theta = \frac{o}{h}$	$\text{cosec}\,\theta = \frac{h}{o}$	$\text{cosec}\,\theta = \frac{1}{\sin\theta}$
$\cos\theta = \frac{a}{h}$	$\sec\theta = \frac{h}{a}$	$\sec\theta = \frac{1}{\cos\theta}$
$\tan\theta = \frac{o}{a}$	$\cot\theta = \frac{a}{o}$	$\cot\theta = \frac{1}{\tan\theta}$

Unit 3

Special triangles

The ratios of certain angles are well known, and we can use them to solve problems. These angles are 0°, 30°, 45°, 60° and 90°.

We can show these angles (except for 0°) in two triangles, as shown below.



The ratios formed by each of these special angles is shown below.

	0°	30°	45°	60°	90°
sin θ	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tan θ	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

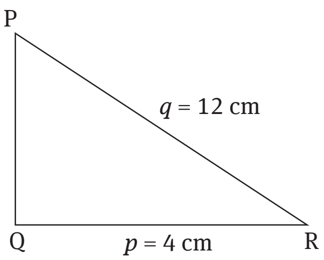
Unit 4

Using your calculator

When all you know is the size of an angle, you can use a calculator to work out the value of a ratio. Your scientific calculator has keys for the trigonometric ratios. Make sure your calculator is in DEG mode when working with degrees.

Note: $\sin^{-1}$ on your calculator is not the same as $\frac{1}{\sin}$. In trigonometry, $\frac{1}{\sin}$ refers to the reciprocal *ratio* for sin, namely cosec. The $\sin^{-1}$ key on the calculator is the reciprocal *function* of sin, which is what you use to calculate the value of an angle when you know the value of the ratio.

The following example shows how to use your calculator to work out the value of a ratio for a particular angle.

Example	**Answer**	**Calculator**
In ΔPQR; ∠Q is 90°, q = 12 cm and p = 4 cm. Find the values of ∠P and ∠R. 	$\sin P = \frac{p}{q}$ $\sin P = \frac{4}{12}$ $\angle P = \sin^{-1}\frac{1}{3}$ ∠P = 19,47° ∠R = 90° – ∠P = 70,53°	sin (1 ÷ 3) =
Note: We name the sides of a right-angled triangle according to the angle names. So for example, the side opposite angle Q is q, and the side opposite angle P is p.		

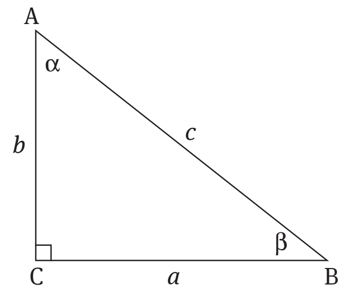
Unit 5

Solving trigonometric equations

5.1 Solve diagrams in two dimensions

Use the following tools to solve right-angled triangles:

- Theorem of Pythagoras: $a^2 + b^2 = c^2$
- Sine: $\sin A = \frac{a}{c}$, $\sin B = \frac{b}{c}$
- Cosine: $\cos A = \frac{b}{c}$, $\cos B = \frac{a}{c}$
- Tangent: $\tan A = \frac{a}{b}$, $\tan B = \frac{b}{a}$



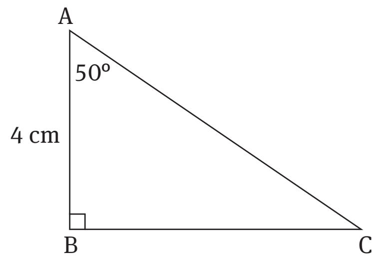
Remember: In triangle ABC, *a* is opposite ∠A, *b* is opposite ∠B, and c opposite ∠C.

You can calculate an unknown side or angle when you know the sizes of:

- an angle and a side
- two sides and we need the angle.

Example

1 Using the diagram alongside, calculate the length of BC if AB is adjacent to ∠A and BC is the opposite side.



$\tan 50° = \frac{BC}{AB}$
BC = AB.tan 50°
BC = 4.tan 50°
BC = 4,77 cm (to 2 decimal numbers)

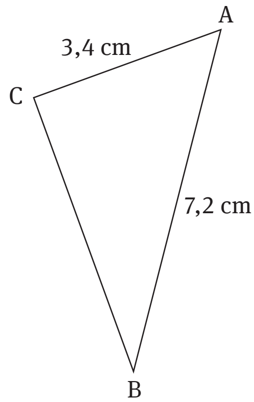
2 Calculate the length of AC, given that AB is adjacent to ∠A and AC is the hypotenuse.

$\cos 50° = \frac{AB}{AC}$
AC.cos 50° = AB
AC = cos 50° = A cos 50°
AC = 6,22 cm (to 2 decimal numbers)

We can also use calculate an angle if we know the value of a ratio.

Example

Using the diagram alongside, calculate the size of ∠B if ∠C = 90°.



AC is the opposite side to ∠B and AB is the hypotenuse:

$\sin B = \frac{AC}{AB}$

$\sin B = \frac{3,4}{7,2}$

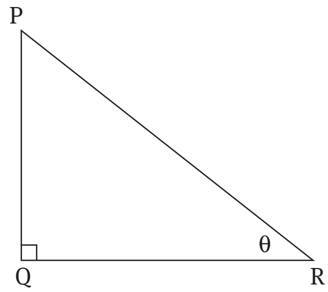
$\angle B = \sin^{-1}\left(\frac{3,4}{7,2}\right)$

Therefore, ∠B = 28,18° (to two decimal places)

Let's work through another example.

Example

In the triangle alongside, $\tan \theta = \frac{5}{12}$.



Calculate the following using the triangle:

1 ∠θ
2 cos θ
3 The length of the hypotenuse

1 $\theta = \tan^{-1} = 22,62°$
2 $\cos \theta = \cos 22,62° = 0,92$
3 The tan ratio is $\frac{\text{opposite}}{\text{adjacent}}$. Therefore, we know that the side opposite θ (PR) is 5 units long, and the adjacent side is 12 units long (QR). Now we can use the theorem of Pythagoras to calculate the length of the hypotenuse, PR:

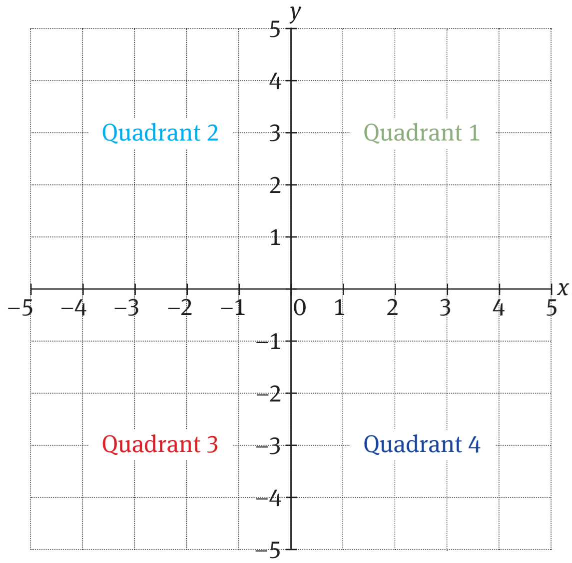
$PR^2 = 5^2 + 12^2 = 25 + 144 = 169$
$PR = \sqrt{169}$
$PR = 13$

Unit 6

Extending the ratios to 0° ≤ θ ≤ 360°

6.1 Trigonometry ratios in four quadrants

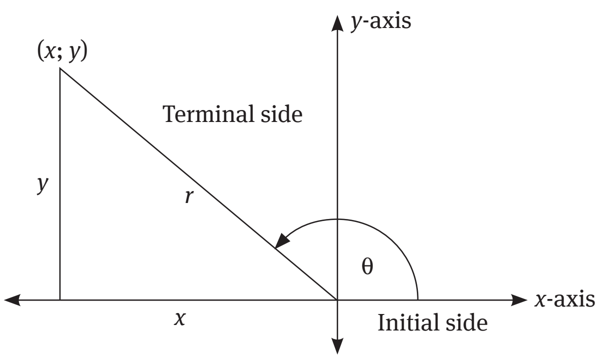
By convention we label the four quadrants of a Cartesian plane as follows:



In trigonometry, we tend to think of an angle as created by a rotating radius (arm). The beginning position of the "arm" is called the **initial side** (usually, the initial side coincides with the positive x-axis; angles with such an initial side are said to be in *standard position*). When the "arm" rotates and ends up in one of the quadrants, the side ("arm") is now called the **terminal side**. The measure of the angle is a number which describes the amount of rotation. If the rotation of the "arm" was clockwise, the angle measure is a negative number; if the rotation was anti-clockwise, the angle measure is positive.

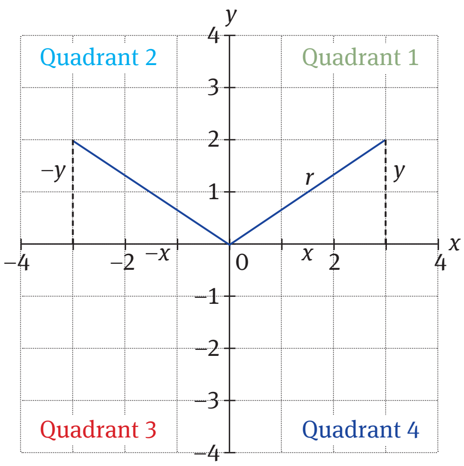
Angles greater than 90°

We **define** the trigonometric functions for angles greater than 90° in the following way:



The trigonometric definitions do not rely on the lengths of the sides of the triangle, but only on the angle. When negative or obtuse angles are used in trigonometric functions, they will sometimes produce negative values. The **CAST diagram** will help you to remember the signs of trigonometric functions for different angles.

The functions will be negative in all quadrants except those that indicate that the function is positive. For example, when the angle is between 0° and 90°, the side *r* is in the first quadrant. All functions will be positive in this quadrant. When the angle is between 90° and 180°, the terminal side is in the second quadrant. This means that only the sine function is positive. All other functions will be negative. When the angle is between 180° and 270°, the terminal side is in the third quadrant. This means that only the tan function is positive. All other functions will be negative. When the angle is between 270° and 360°, the terminal side is in the fourth quadrant. This means that only the cosine function is positive. All other functions will be negative.



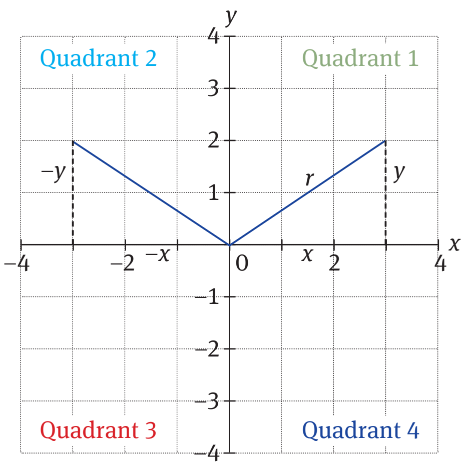
Quadrant 1

sine of an angle = $\frac{y}{r}$

cosine of an angle = $\frac{x}{r}$

tan of an angle = $\frac{y}{x}$

All positive, because x, y and $r > 0$



Quadrant 2

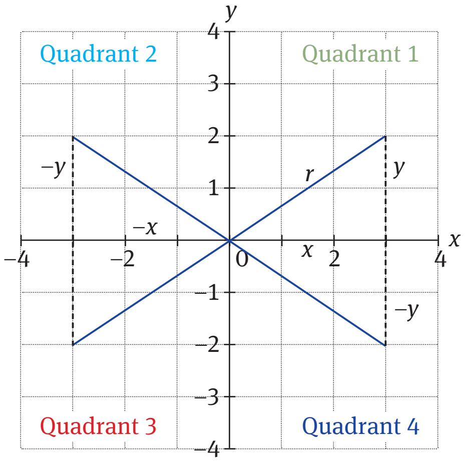
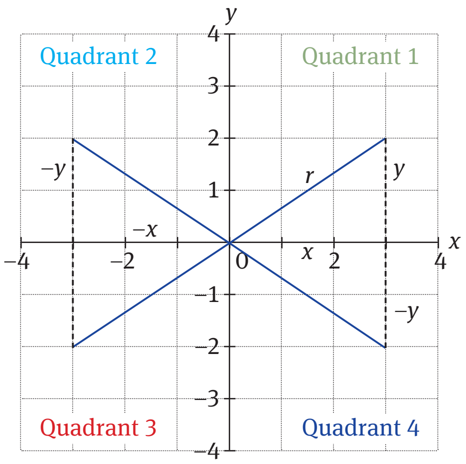
sine of an angle = $\frac{y}{r}$

cosine of an angle = $\frac{-x}{r}$

tan of an angle = $\frac{y}{-x}$

Because x is negative in the second quadrant, all the functions containing x will be negative, i.e. cos and tan.

Only the sine function is positive, because y and $r > 0$.

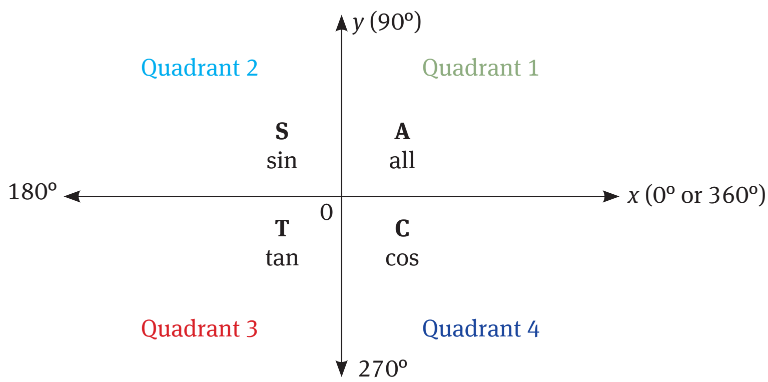
	**Quadrant 3** sine of an angle = $\frac{-y}{r}$ cosine of an angle = $\frac{-x}{r}$ tan of an angle = $\frac{-y}{-x} = \frac{y}{x}$ Only the **tan** function is positive in the third quadrant (neg ÷ neg = pos).
	**Quadrant 4** sine of an angle = $\frac{-y}{r}$ cosine of an angle = $\frac{x}{r}$ tan of an angle = $\frac{-y}{x}$ In quadrant 4 only the cos function is positive, because x and $r > 0$.

The CAST rule

There is a simple rule by which you can remember all of these results. Notice in the first quadrant, **A**ll the functions are positive; in the second quadrant, only the **S**ine is positive; in the third, only the **T**angent is positive; and in the fourth quadrant, only the **C**osine is positive.

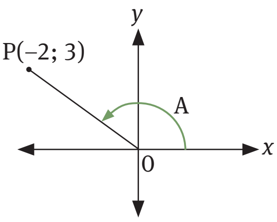
This is called the **CAST Rule** and tells you which function is **positive** in each quadrant. And the C, A, S and T stand for **cosine**, **all**, **sine** and **tangent**.

S	A
T	C



Examples using the CAST rule

1. Without using a calculator, use the figure to calculate the following:
   1.1 OP
   1.2 tan A
   1.3 cos A
   1.4 $\sin^2 A - \cos^2 A$



Notice that the point P is in the SECOND quadrant, only sine is positive in the second quadrant.

S	A
T	C

1.1 Use Pythagoras' theorem to calculate the length of OP:
$OP^2 = 3^3 + (-2)^2$
$\therefore OP = \sqrt{9+4} = \sqrt{13}$

1.2 $\tan A = \frac{y}{x} = \frac{3}{-2} = -\frac{3}{2}$
(tan is negative in the 2nd quadrant!)

1.3 $\cos A = \frac{x}{r} = \frac{-2}{\sqrt{13}}$
(cos is also negative in the 2nd quadrant!)

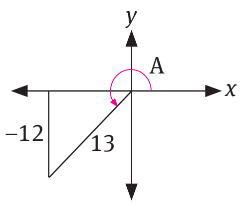
1.4 $\sin A = \frac{y}{r} = \frac{3}{\sqrt{13}}$
therefore, $\sin^2 A - \cos^2 A$
$= \frac{4}{13} + \frac{9}{13}$
$= \frac{13}{13}$
$= 1$

2. If $\sin A = -\frac{12}{13}$ and $\hat{A} \in [90°; 270°]$ determine, without the use of a calculator, the values of
   2.1 cos A
   2.2 tan A

Notice that the angle A could be in the second or third quadrants, but sine is negative in the third quadrant. So, angle A has to be in the THIRD quadrant.

S	A
T	C

It is useful to make a drawing:



The x-value = −5 (Pythagoras)

2.1 $\cos A = \frac{x}{r} = \frac{-5}{13}$ (cos is negative in the 3rd quadrant!)

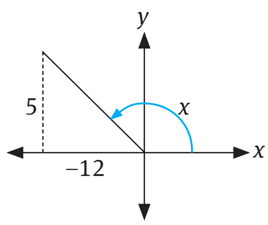
2.2 $\tan A = \frac{y}{x} = \frac{-12}{-5} = \frac{12}{5}$ (tan is positive in the 3rd quadrant!)

3. If $\tan x = -\frac{5}{12}$, $0° < x < 180°$, use a sketch (no calculator) to determine the value of $3 \sin x - 2 \cos x$.

Notice that the tan function is negative and the given domain covers the first two quadrants. Tan is negative in the second quadrant.

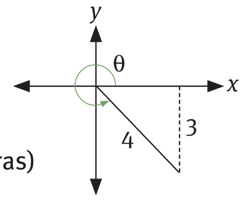
S	A
T	C

The terminal side: 13 *(Pythagoras)*



Sine is positive in the 2nd quadrant and cosine is negative.

$3 \sin x - 2 \cos x$
$= 3\left(\frac{5}{13}\right) - 2\left(\frac{-12}{13}\right)$
$= \frac{15}{13} + \frac{24}{13}$
$= \frac{39}{13}$
$= 3$

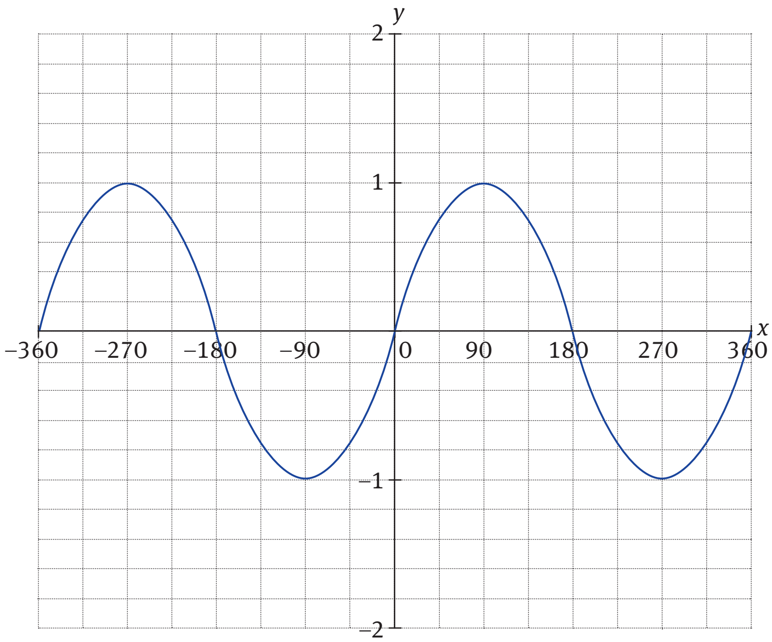
4. If 4 sin θ + 3 = 0 and 0 › 270° determine, without using a calculator: 4.1 tan θ 4.2 $\cos^2\theta$ *Notice that angle is greater than 270°. So, the operating quadrant must be the fourth quadrant and the sine function is also negative in the fourth quadrant.* **S** | **A** **T** | **C**	4 sin θ + 3 = 0 $\therefore \sin\theta = -\frac{3}{4}$  $x = \sqrt{4^2 - (-3)^2}$ $= \sqrt{16 - 9}$ (Pythagoras) $= \sqrt{7}$ 4.1 $\tan\theta = -\frac{3}{\sqrt{7}}$ (tan is negative in the 4th quadrant) 4.2 $\cos 20 = \left(\frac{\sqrt{7}}{4}\right)^2 = \frac{7}{16}$ (cos is positive in the 4th quadrant)

Unit 7

Graphs of the trigonometric functions

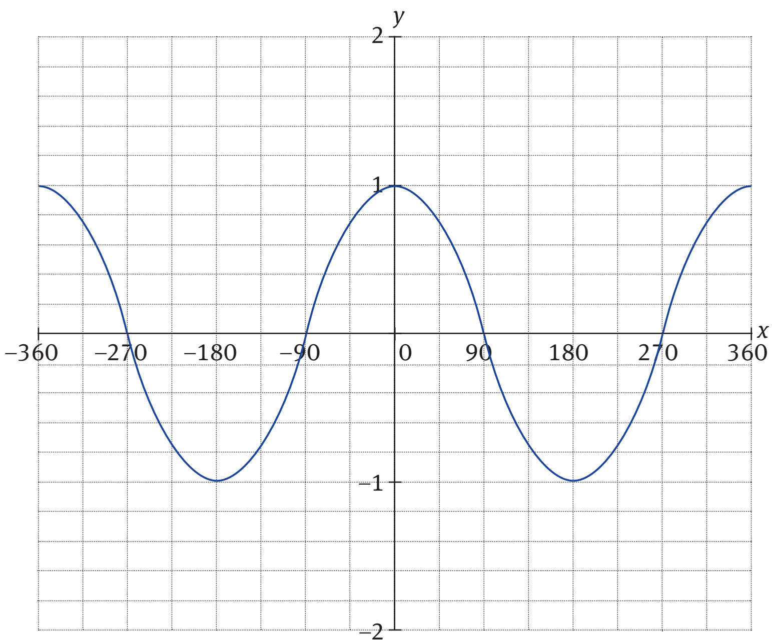
7.1 The sine and cosine graphs

1. $y = \sin\theta$



The graph repeats itself every 360°
Domain: All real numbers
Range: [–1, 1]
Maximum value: 1 when θ = –270°; 90°
Minimum value: –1 when θ = –90°; 270°
Zero: when the graph cuts the θ-axis, i.e. when θ = –360°; –180°; 0°; 180°; 360°

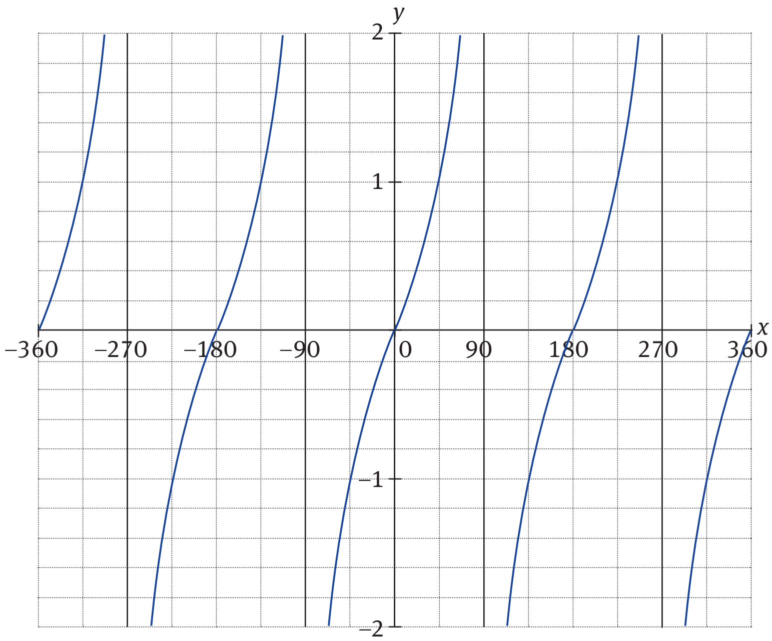
2. $y = \cos\theta$



The graph repeats itself every 360°
Domain: All real numbers
Range: [–1, 1]
Maximum value: 1 when θ = –360°; 0°; 360°
Minimum value: –1 when θ = –180°; 180°
Zero: when the graph cuts the θ-axis, i.e. when θ = –270°; –90°; 90°; 270°

7.2 The tan graph

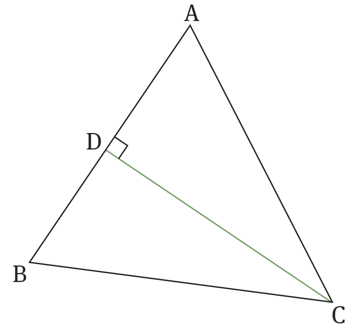
1. $y = \tan\theta$



The graph repeats itself every 180°
No maximum or minimum turning points
Zero: when the graph cuts the θ-axis, i.e. when θ = –360°; –180°; 0°; 180°; 360°
Asymptotes: at θ = –270°; –90°; 90°; 270°

Questions

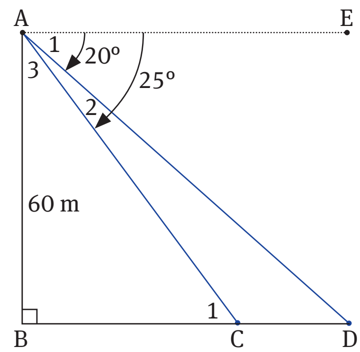
1 In ΔABC, DC is perpendicular to AB, $\angle A = 55°$, AD = BD and DC = 15 cm.
(Use your calculator to answer following questions.)



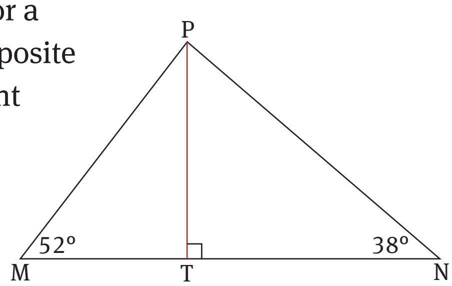
a Find the length of AD and then AB.
b Find the length of AC.
c Find the length of BC. (What does you answer imply about ΔABC?)
d Find $\angle DBC$.

2 A rocket is launched vertically and changes direction through an angle of 55° to the vertical line after 2 minutes. If the average speed of the rocket is 6 000 km/h, calculate:
a The height of the rocket after 2 minutes.
b The height of the rocket after 3 minutes.
c The horizontal distance from A, the launching tower, to B, a point directly under the rocket after 3 minutes.

3 From the top of a perpendicular cliff, AB, 60 m above sea level, the angles of depression to two boats at C and D, in the same vertical plane as B, are 20° and 25°, respectively.
Calculate the distance between the two boats at C and D (*correct to one decimal place*).



4 In the figure, M and N represent service vehicles for a cellular telephone company and are parked on opposite sides of a cellular phone mast, PT. MTN is a straight line. The angles of elevation of mast PT from the vehicles M and N are 52° and 38° respectively.
If MTN = 160 m, calculate:



a the magnitude (size) of $\angle MPN$
b the length of MP, to the nearest metre
c the height of the mast PT, to the nearest metre

5 Use your calculator to solve the following equations, correct to two decimal numbers ($0° \leq x \leq 90°$).

a $\sin x = 0,57$

b $3\cos x = 0,96$

c $\frac{2}{3}\tan x = 4,2$

d $\sin (x + 25°) = 0,813$

e $\tan x = \sin 42°$

6 Using the special angles, solve the following equations for $0° \leq x \leq 90°$.

a $2\sin x = \sqrt{3}$

b $\cos 3x = \frac{1}{2}$

c $4\sin (2x - 10°) - 1 = 1$

d $x.\tan 60° = \cos 30°$

Chapter 6

Functions

Overview

In this chapter, you will learn how to:

- represent functions using function notation: $f(x)$
- graph the basic functions $f(x) = ax^2 + b$, $f(x) = \frac{k}{x}$, $f(x) = b^x$
- graph transformations of the basic functions, including vertical *shifts*, *stretches*, and *shrinks*, as well as reflections across the x- and y-axes
- investigate and explain the characteristics of a function: *domain*, *range*, *zeros*, *intercepts*, *intervals of increase and decrease*, *maximum and minimum values* and *end behaviour*
- relate to a given context the *characteristics of a function*, and use graphs and tables to investigate its behaviour.

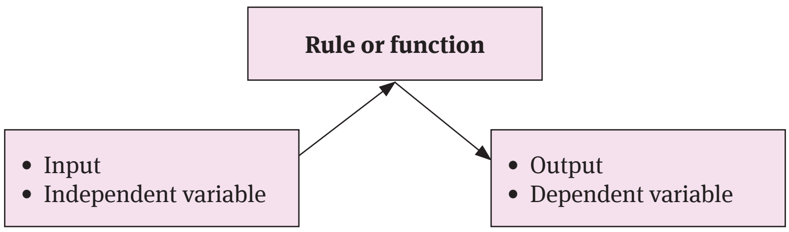
Unit 1

What is a function?

1.1 Dependent and independent variables

A function is like a machine:

- it has an input and an output
- there is a relationship between input and output values
- the output depends on the input, so we say that the output is a dependent variable and the input is the independent variable.
- "$f(x)$ = …" is the classic way of writing a function.



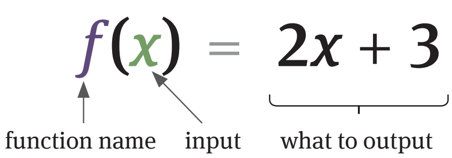
1.2 Relationship between variables

- A function relates inputs to outputs
- A function is a special type of relation in which:
  - the output value is dependent on the input value
  - any input produces only one output (not this or that)
  - an input and its matching output are together called an ordered pair
  - a function can also be seen as a set of ordered pairs

For example: $y = 2x + 3$ is a function.

First, it is useful to give a function a **name**. The most common name is "f", but you can have other names, such as "g". But f is used most often in mathematics.

Input	Relationship	Output
0	×2 + 3	3
1	×2 + 3	5
2	×2 + 3	7
3	×2 + 3	9
…	…	…

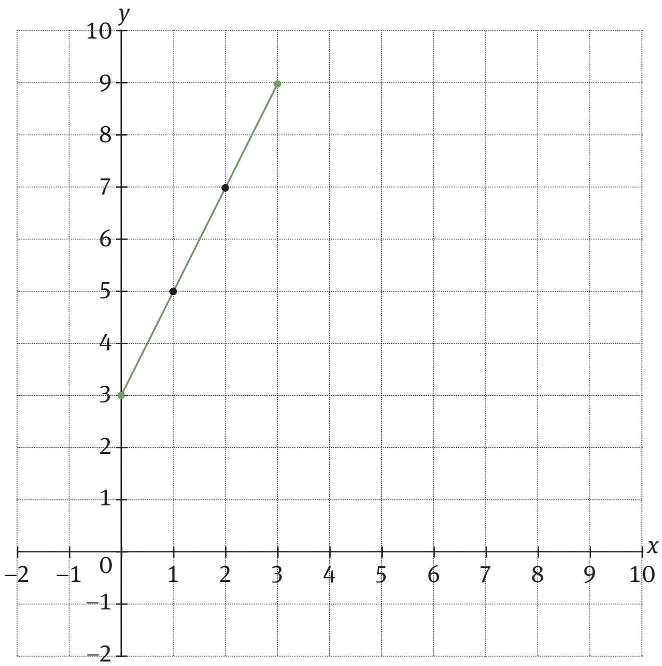


We read the function statement as: *"f of x equals two x plus three"*.

What goes into the function is put inside brackets ( ) after the name of the function. So, $f(x)$ shows you the function is called "f", and "$2x + 3$" tells you that function "f" takes "x", *multiplies it by two and adds three*.

The answer is the output. The output value **uniquely** depends on the input value. An input of 3 gives an output of 9: $f(3) = 9$.

The input and output values can be plotted as coordinates on a graph, in the form (input; output) or $(x; f(x))$: (0; 3), (1; 5), (2; 7), (3; 9), etc.



So, when plotted in the Cartesian plane, the function $y = 2x + 3$ forms a straight line of the kind $y = ax + q$, also written as $f(x) = ax + q$.

Example

A function f is defined by $f(x) = 2 + x - x^2$. What is the value of $f(-3)$?

$f(-3) = 2 + (-3) -(-3)^2 = 2 - 3 - 9 = -10$

Unit 2

Graphs of functions

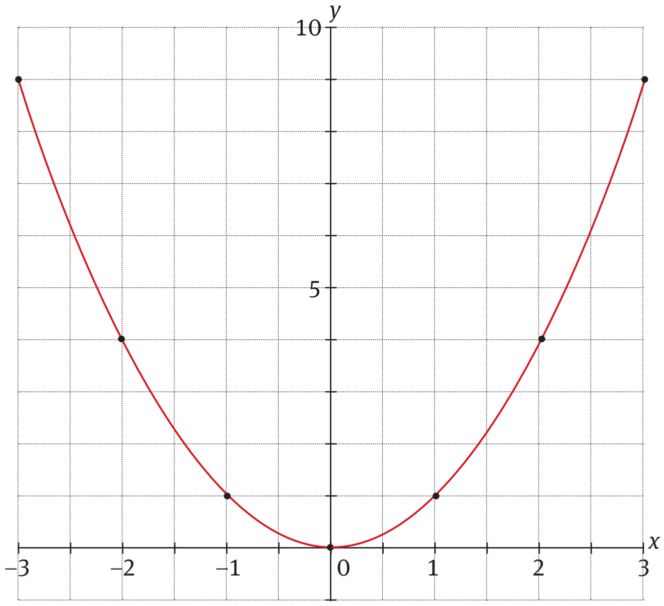
At first, it may be helpful to generate a table of values using numbers from the domain (x-values).

Your x-values should include examples of negative numbers and fractions.

2.1 The graph of the parabola, $y = x^2$

- The graph forms a parabola
- Axis of symmetry: $y = 0$ (y-axis)
- Turning point at (0; 0)
- Intercept with x-axis at $x = 0$ and intercept with y-axis at $y = 0$

x (**domain**)	–2	–1	0	1	2
y (**range**)	4	1	0	1	4

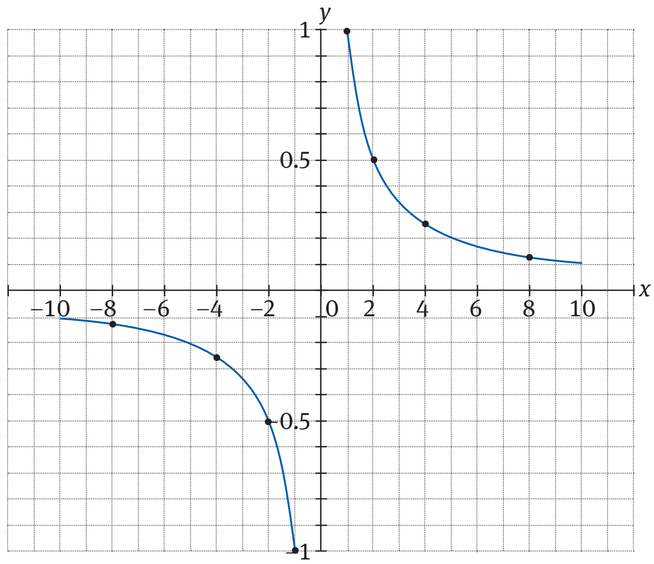


Equation 1: $y = x^2$

2.2 The graph of $y = \frac{1}{x}$

- The graph forms a hyperbola
- Axis of symmetry: $y = x$
- Intercept with y-axis at $y = 1$

x (domain)	−2	−1	0	1	2
y (range)	−0,5	−1	undefined	1	0,5

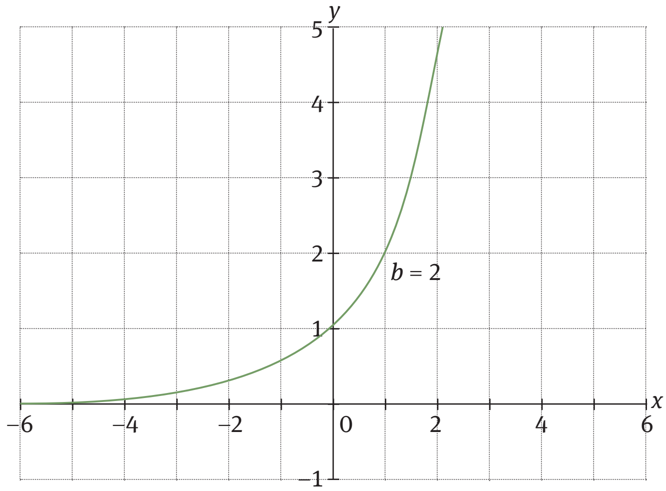


Equation 1: $y = \frac{1}{x}$

2.3 The graph of $y = b^x$, $b > 0$ and $b \neq 1$

- The graph forms an exponential graph
- Axis of symmetry: $y = x$
- Intercept with y-axis at $y = 1$

x (domain)	−2	−1	0	1	2
y (range)	0,25	0,5	1	2	4



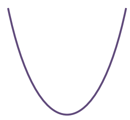
Equation 1: $y = 2^x$

Unit 3

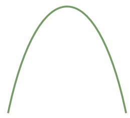
The graph of $y = ax^2 + q$

3.1 The effect of *a*

- The effect of a is a vertical stretch and/or a reflection about the x-axis.
- The value of a gives an indication of the steepness (wide or narrow) of the two "arms".
- For $a > 0$:
  - The parabola will have a minimum value.
  - The "arms" point upwards.
  - Range is $y \geq q, y \in$ R

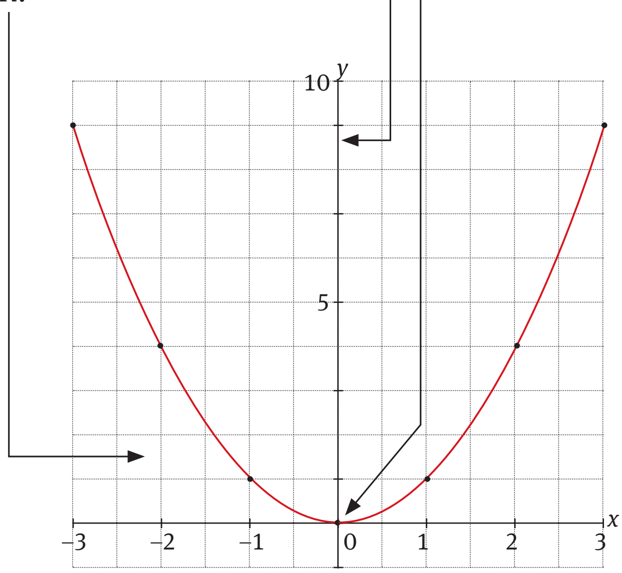


- For $a < 0$:
  - The parabola will have a maximum value.
  - The "arms" point downwards.
  - Range is $y \leq q, y \in$ R



3.2 The effect of *q*

- The y-intercept (where the graph cuts the y-axis) is equal to the value of q or $y = q$ (this is also the maximum or minimum value).
- The effect of q is a vertical shift.
- The turning point of the graph is at $(0; q)$.
- The axis of symmetry is the y-axis ($x = 0$).
- Domain is $x \in R$.

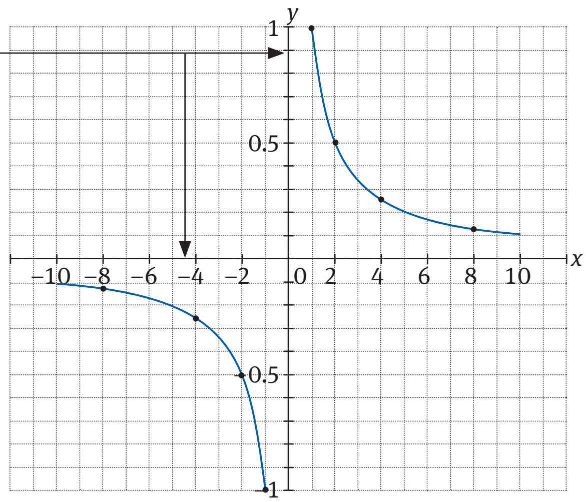


Unit 4

The graph of $y = \frac{a}{x} + x$

4.1 The effect of *a*

- Graphs that represent the equation $y = \frac{a}{x} + q$ form a hyperbola.
- We call the *x*- and *y*-axes the asymptotes.
    - The line $x = 0$ (*y*-axis) is called a vertical asymptote of the function if *y* approaches infinity (positive or negative) as *x* approaches 0.
    - The line $y = 0$ (parallel to *x*-axis) is called a horizontal asymptote of the function.
- For $a > 0$, the graph lies in the 1st and 4th quadrants
- For $a < 0$, the graph lies in the 2nd and 3rd quadrants.



4.2 The effect of *q*

- For $q = 0$, the graph will never intercept the *x*- or *y*-axis because division by 0 is undefined.
- For $q \neq 0$, the line $y = q$ is the horizontal asymptote and $x = 0$ the vertical asymptote.
- Domain: all real *x*-values, except 0
- Range: all real *y*-values, except $y = q$

Example

Sketch the graph of $y = \frac{6}{x} - 1$

Domain: $x \in$ R, $x \neq 0$

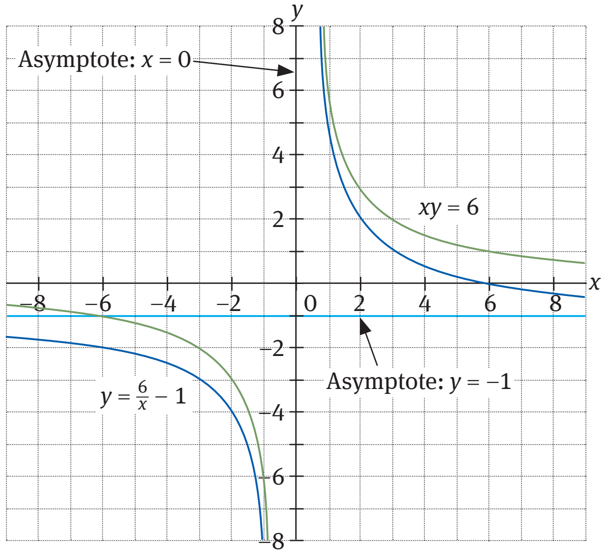
Range: $y \in$ R, $y \neq 1$

x-intercept (let $y = 0$): $x = 6$

y-intercept: none

Asymptotes:

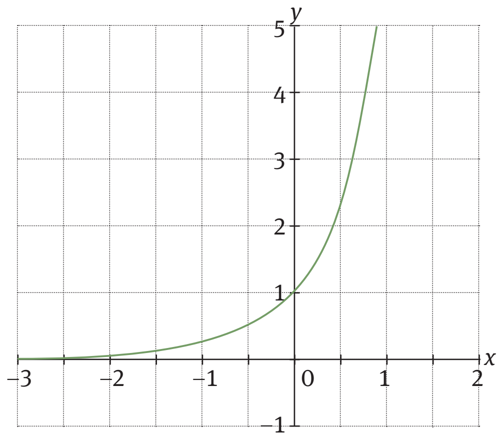
- The graph is only decreasing, for $x \in$ R, $x \neq 0$
- The graph shifts down one unit: new asymptote, $y = -1$
- Therefore, the asymptotes are: *y*-axis and $y = -1$


y
x
Asymptote: x = 0
Asymptote: y = −1
xy = 6
y = 6/x − 1
8
6
4
2
0
−2
−4
−6
−8
2
4
6
8
−8
−6
−4
−2

Unit 5

The graph of $y = ab^x + q$, $b > 0$ and $b \neq 1$

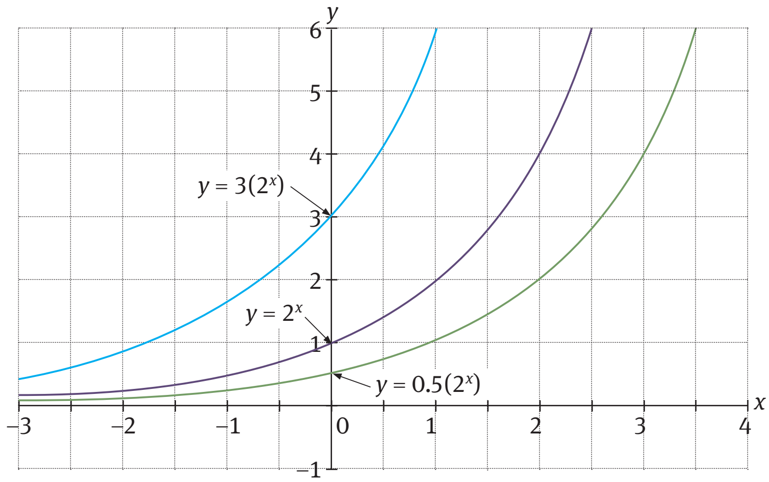
5.1 Possible values of *b*



- Graphs that represent the equation $y = ab^x + q$ form an exponential graph in two quadrants of the Cartesian plane.
- When $b > 1$ the y-values increase as x increases and the function is called an increasing function.
- The graph will incline to the right.
- When $0 < b < 1$ the y values will decrease and the function is a decreasing function.
- The graph will decline to the right.
- The y-intercept (where the graph cuts the y-axis) is equal to the value of a for $q = 0$ and is therefore $a + q$ for any other value.
- There are no x-intercepts.
- The line $y = q$ is the asymptote for the graph.
- Domain: $x \in R$
- Range: $y > 0, x \in R$

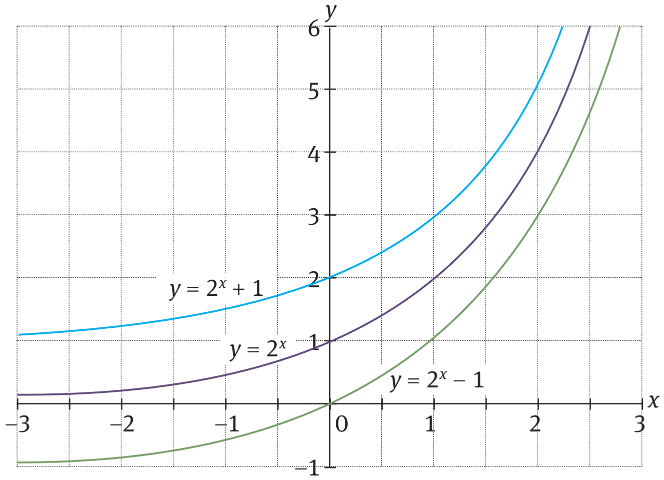
5.2 The effect of *a*

- a affects the steepness of the graph
- y-intercept: $y = a$
- If $a < 0$, then the graph reflects about the y-axis



5.3 The effect of q

- The value of q shifts the graph up or down (vertical shift)
- y-intercept: $1 + q$



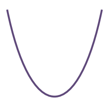
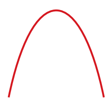
Unit 6

Sketching graphs

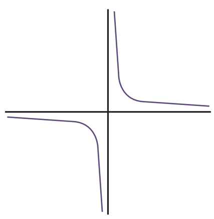
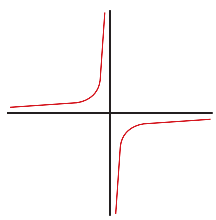
6.1 The parabola

To plot graph of any quadratic function (parabola), we need answers to these questions:

- What is the sign of a (the coefficient of x^2 in quadratic function)?
- Does the graph of quadratic function intersect the x-axis? And if it does, at what point does it intersect?
- Does the graph of quadratic function intersect the y-axis?
- What is the maximum or minimum value of function?

For $a > 0$: The parabola will have a minimum value. The "arms" point upwards. 	For $a < 0$: The parabola will have a maximum value. The "arms" point downwards. 
x-intercept(s): Let $y = 0$ and solve for x	y-intercept(s): Let $x = 0$ and solve for y
Because $y = ax^2 + q$ has no term in x, the axis of symmetry is the y-axis.	The turning point of the parabola is $(0; q)$

6.2 The hyperbola

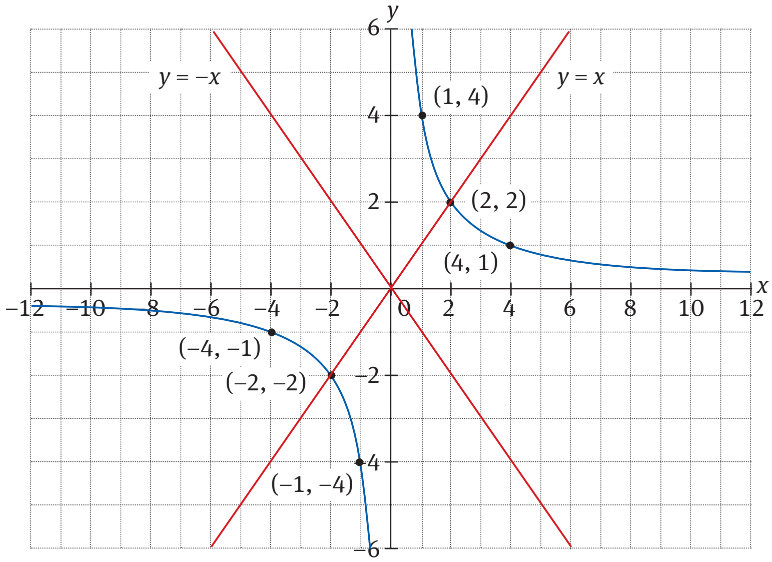
$a > 0$ 	$a < 0$ 
x-intercepts: None, except when $q \neq 0$. Then put $y = 0$ and solve for x	y-intercepts: None, the y-axis is an asymptote
Asymptotes: The y-axis ($x = 0$) The 2nd asymptote is the line $y = q$	
The closest points to the axes: $(\sqrt{a}; \sqrt{a} + q)$ and $(-\sqrt{a}; -\sqrt{a} + q)$	Domain: $x \in R; x \neq 0$ Range: $y \in R; y \neq q$

Example

Given $y = \frac{4}{x}$, sketch the graph of the function.

- Show the coordinates where $x = 1$, $x = -1$, $x = 4$ and $x = -4$ clearly on your graph. To do so, work out the y-values and plot the points: $(1; 4)$, $(-1; -4)$, $(4; 1)$, $(-4; -1)$

- Draw two axes of symmetry on your sketch and give their equations.
  ($y = x$ and $y = -x$)
- Find the point(s) of intersection of the hyperbola and the axes of symmetry.
  $(\sqrt{a}; \sqrt{a} + q)$ and $(-\sqrt{a}; -\sqrt{a} + q) = (-2; -2)$ and $(2; 2)$
- The graph will be in quadrants 1 and 3, because $a > 0$
- The domain is $x \in$ R, $x \neq 0$
- The range is $y \in$ R, $y \neq q$



6.3 The exponential function

Use the characteristics of the graph to determine the general shape.

a	*b*	**Characteristic**
$a < 0$	$0 < b < 1$ (a positive fraction)	decreasing
$a > 0$	$0 < b < 1$ (a positive fraction)	Increasing
$a < 0$	$b > 1$	Decreasing
$a > 0$	$b > 1$	Increasing

- If $a < 0$, the graph is below the asymptote and if $a > 0$, the graph is above the asymptote.
- asymptote: $y = q$
- y-intercept: $y = a + q$
- Domain: $x \in$ R
- Range if $a < 0$: $y < q$, $y \in$ R
- Range if $a > 0$: $y > q$, $y \in$ R

Example

Given: $f(x) = 3$, sketch the graph of f. Show any intercepts with the axes.

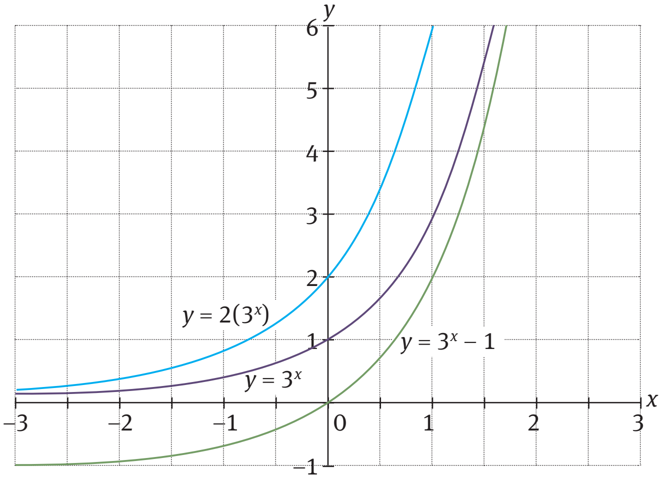
General form: $y = ab^x + q$

$a > 0$ (=1)	$b > 1$ (=3)	Increasing

Therefore:

- Asymptote: $y = 0$ ($q = 0$)
- y-intercept: $y = 1$ ($a = 1$)
- Domain: $x \in$ R
- Range if $a > 0$: $y > q$, $y \in$ R

On the same system of axes, sketch the graph of $g(x) = 2.3^x$. Clearly show any intercepts with the axes.

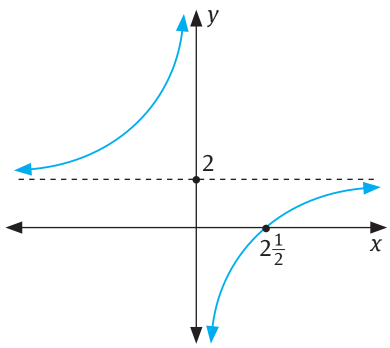


Unit 7

Finding the equations of graphs

To find the equation of a graph, use the general form of the equation and the characteristics of the graph to write down what you know. Then, substitute point(s) from the graph into the equation to find the values of any unknown variables.

Example



The graph alongside is the graph of $y = \frac{a}{x} + q$.

1 Find the equation of the graph.

2 Give a reason why $y = 2$ is an asymptote.

1 The asymptote is at $y = 2$, so we know $q = 2$.
We can then use the point (2; 0) in the general equation to find a:

$0 = \frac{a}{\frac{5}{2}} + 2$

$\frac{a}{\frac{5}{2}} = -2$

$a = -2 \times \frac{5}{2}$

$a = -5$

Therefore, $y = \frac{-5}{x} + 2$

The range of the graph is $y \in R; y \neq 2$

2 If we make x the subject of the equation, we have:

$y = \frac{-5}{x} + 2$

$xy = -5 + 2x$

$x(y - 2) = -5$

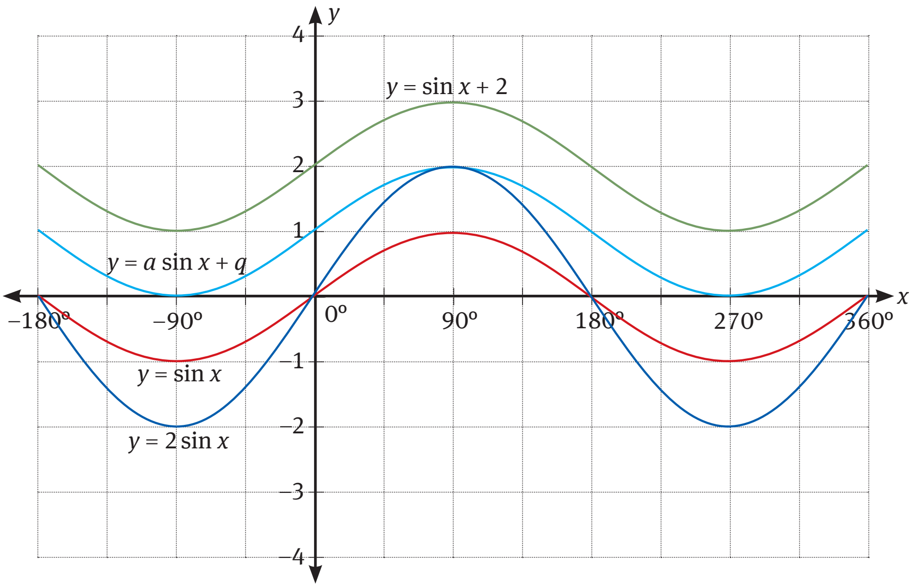
$x = \frac{-5}{y - 2}$

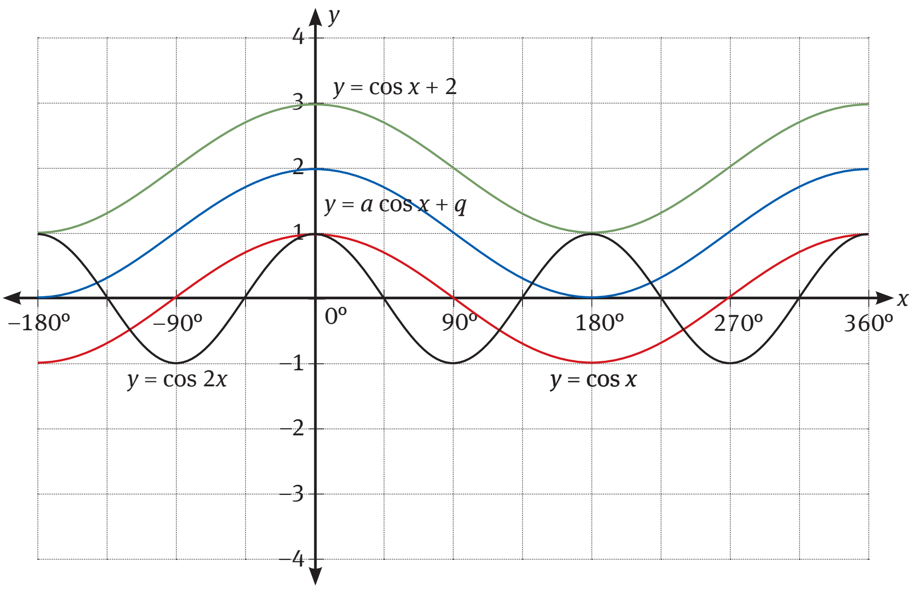
Therefore, we cannot have $y = 2$, because division by 0 is undefined.

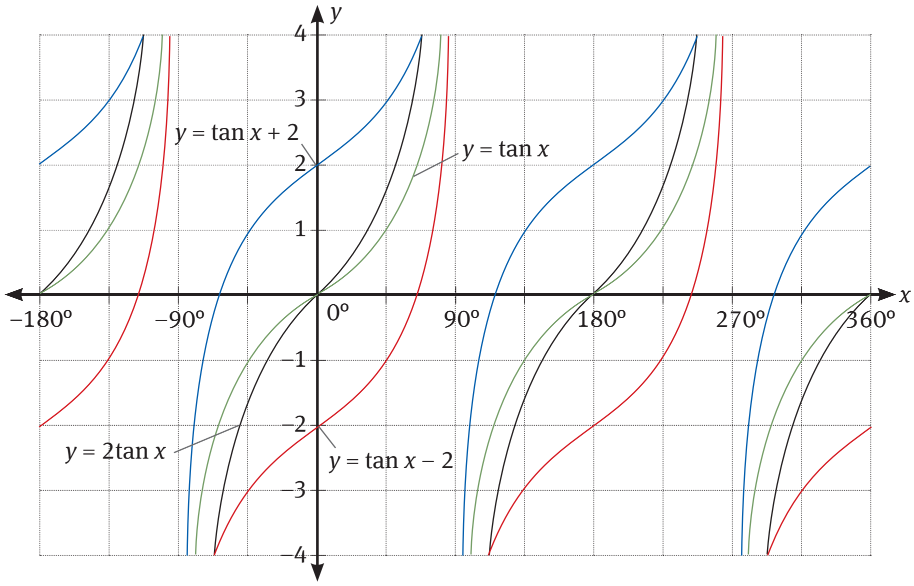
Unit 8

The effect of a and q on trigonometric graphs

- The trigonometric functions form periodic graphs
- $a.\sin x + q$ and $a.\cos x + q$ have the form of a repeating wave with period 360º
- For $q > 0$, the graph shifts q units up, and for $q < 0$, it shifts q units down.
- The variable a stretches the graphs of the trigonometric functions.
- Multiplying $\sin x$ or $\cos x$ by a positive constant, a, increases the amplitude to the value of a






y
4
3
2
1
−1
−2
−3
−4
x
−180°
−90°
0°
90°
180°
270°
360°
y = tan x + 2
y = tan x
y = 2tan x
y = tan x − 2

Questions

1 Given the equation $y = x^2 - 4$:
   - a Describe the form of the graph.
   - b Does the graph have a minimum or maximum value? Give a reason for your answer.
   - c What is the y-value for the minimum or maximum?
   - d For which values of x will the graph intercept the x-axis?
   - e Sketch the graph of the function.
   - f Write down the equation if the graph moves up 4 units.
   - g What will the x-intercept(s) be of the new graph?
   - h Write down the domain and range of the function $y = x^2 - 4$ in interval notation.
   - i Write down the equation of the axis of symmetry.

2 Given a parabola with turning point (0; 3) and x-intercept (−3; 0):
   - a Write down the equation of the graph.
   - b Write down the domain and range of the graph.
   - c Write down the equation of the axis of symmetry of the graph.

Chapter 7

Polygons

Overview

In this chapter, we will first revise what you have learnt previously about polygons and the concepts of similarity and congruence in triangles. Then, we extend this knowledge to quadrilaterals, including the kite, parallelogram, rectangle, rhombus, square and trapezium. Here, you will investigate and learn about some of the important geometric properties of each of these shapes.

Here, you will learn:

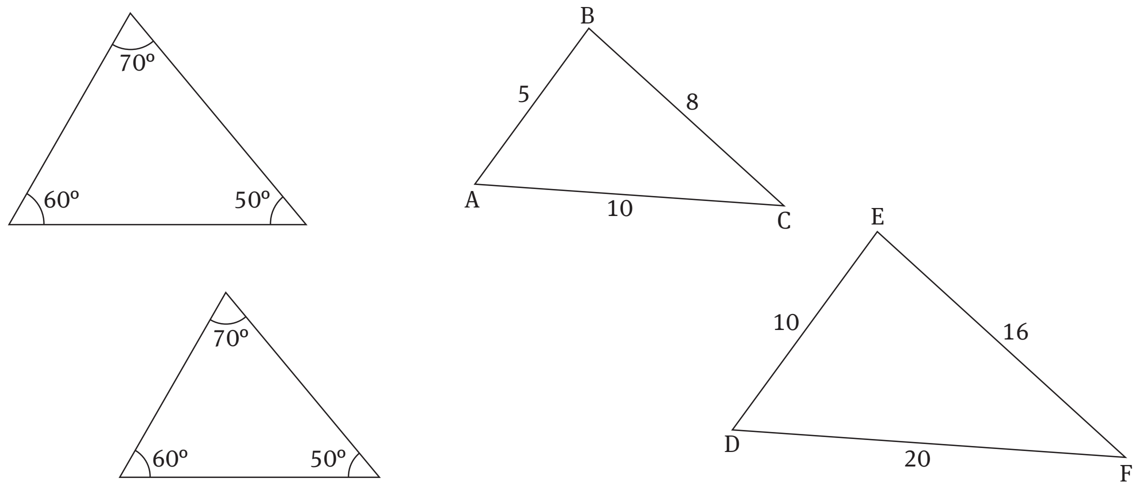
- how two triangles are similar if they are the same shape, but different sizes
- about corresponding angles and sides
- how two triangles are congruent if their corresponding sides and angles are the same size
- about different types of quadrilaterals (trapezium, parallelogram, rhombus, rectangle, square, kite)

Unit 1

Similar triangles

Similar figures have the same shape, but are different sizes. In the case of triangles:

- Corresponding angles are equal.
- Corresponding lengths are in the same ratio.



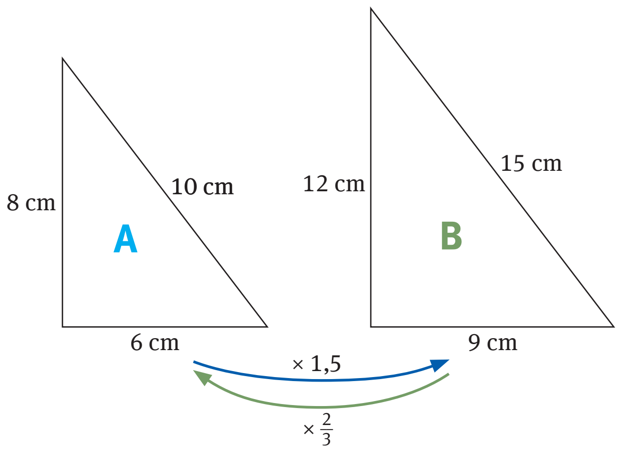
There are three ways to determine if two triangles are similar.

1. **AAA similarity**: If in two triangles, the corresponding angles are equal, the triangles are similar. (The 3rd set of angles will be equal, because of the sum of the interior angles of a triangle being 180°.)
2. **SSS similarity**: If the corresponding sides of two triangles are proportional the triangles are similar.
3. **SAS similarity**: If one angle of a triangle is equal to one angle of the other triangle and the sides containing these angles are proportional, the triangles are similar.

Finding the scale factor

We use the scale factor to find missing lengths of similar figures. Here, we can:

- Find a corresponding side in each shape when we know the length of both.
- Divide the length in the shape we are going to by the length in the shape we are coming from.



Example

To find the scale factor from A to B, divide the length in B by the length in A:
$9 \div 6 = 1{,}5$
The scale factor from A to B is 1,5.

Similarly, to find the scale factor from B to A, divide the length in A by the length in B:
$6 \div 9 = \frac{2}{3}$
The scale factor from B to A is $\frac{2}{3}$.

To prove that two triangles are similar, we have to show that **one** (not all) of the following statements is true:

- The three sides are in the same proportion.
- Two sides are in the same proportion, and their included angle is equal.
- The three angles of the first triangle are equal to the three angles of the second triangle.

Notes:

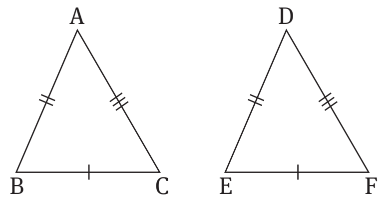
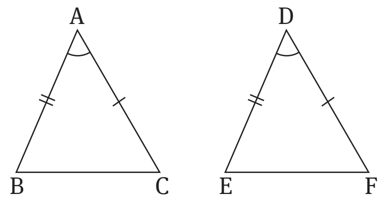
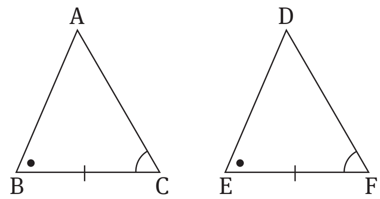
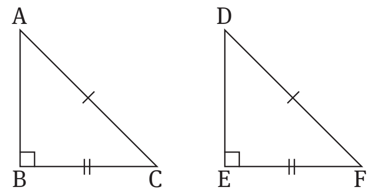
- The symbol for similarity is |||
- The order of the points in the names of the triangles is important. Equal angles of the two triangles must coincide.

Unit 2

Congruent triangles

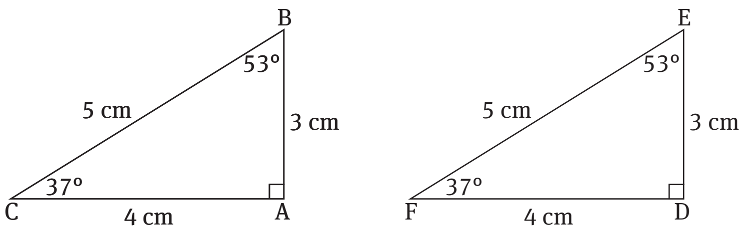
Congruent triangles have the same size *and* the same shape. The corresponding sides and the corresponding angles of congruent triangles are equal.

There are four methods to prove if two triangles are congruent.

1. The Side-Side-Side principle (SSS): The three corresponding sides of two triangles must be the same.	
2. The Side-Angle-Side principle (SAS): The two corresponding sides and the included angle of two triangles must be equal.	
3. The Angle-Side-Angle principle (ASA): The two corresponding angles and the included side of two triangles must be equal.	
4. The Right-Angle-Hypotenuse-Side principle (RHS): The hypotenuse and one corresponding side of the two right-angled triangles must be equal.	

Example

$\Delta ABC \equiv \Delta DFE$ because:



$\angle A = \angle D = 90°$
$\angle B = \angle E = 53°$
$\angle C = \angle F = 37°$

Notes:

- The symbol for congruency is ≡
- The order of the points in the names of the triangles is important. Equal angles of the two triangles must coincide.

Unit 3

Quadrilaterals

- A polygon with four sides is called a quadrilateral.
- The special types of quadrilaterals include the parallelogram, rectangle, rhombus, square, trapezoid and kite.
- It is important to understand the relationship of these figures and their properties in order to properly classify or identify a figure.
- A conjecture is an assertion that is likely to be true but has not been formally proven.

- To prove conjectures you need to complete a proof that will always be true.
- To show that a conjecture is false, show one counter example.

3.1 Parallelogram

Properties:
- opposite sides are parallel
- opposite angles are congruent
- opposite sides are congruent
- diagonals bisect each other
- consecutive angles are supplementary

Theorems:
1. If one pair of opposite sides of a quadrilateral is equal and parallel, then the quadrilateral is a parallelogram.
2. If both pairs of opposite sides of a quadrilateral are equal, then the quadrilateral is a parallelogram.
3. If both pairs of opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.
4. If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

3.2 Rectangles

Properties:
- has all the properties of a parallelogram
- diagonals are congruent
- contains four right angles

Theorem:
1. If the diagonals of a parallelogram are equal, then the parallelogram is a rectangle.

3.3 Rhombus

Properties:
- has all the properties of a parallelogram
- four sides are equal in length
- diagonals are perpendicular
- diagonals bisect each pair of opposite angles

Theorems:
1. If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.
2. If each diagonal of a parallelogram bisects a pair of opposite angles, then the parallelogram is a rhombus.

3.4 Square

Properties:
- has all the properties of a parallelogram
- diagonals are congruent and perpendicular
- is a rectangle with all sides congruent
- is a rhombus with four right angles

3.5 Kite

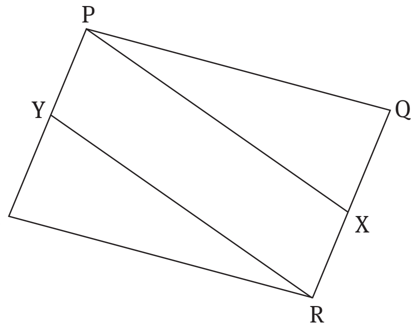
Properties:
- is a quadrilateral that has exactly two distinct pairs of adjacent congruent sides

3.6 Trapezium

Properties:
- one pair of opposite sides that are parallel
- two parallel sides are called bases and the nonparallel sides are the legs
- isosceles trapezoid has one pair of congruent sides and congruent diagonals

Example

1 PQRS is a parallelogram with SY = QX.
Prove that PXRY is a parallelogram.

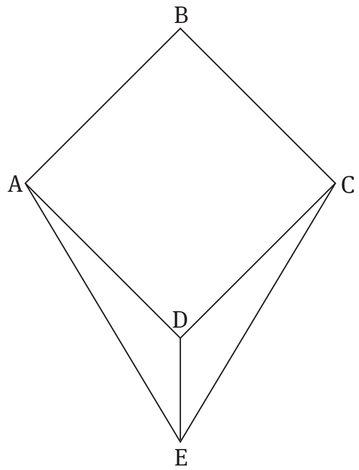


Steps to follow:
- Identify what is given and what to prove.
- Redraw the diagram, filling in the information given.
- Plan your proof using the drawing.
- Write up the proof explaining your reasoning using symbols and valid reasons in brackets.

QR = PS	(opp sides of parallelogram)
QX = YS = PY = XR	
And PY||XR	(given)

Therefore, PYRX is a parallelogram (one pair of opp. sides || and =)

2 ABCD is a rhombus and AE = CE.
Prove that ΔADE = ΔCDE.

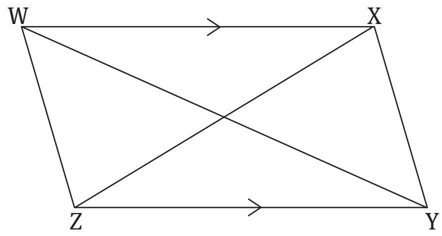


Use congruency to prove the required angles are equal. Remember, a rhombus is a parallelogram with all sides equal.

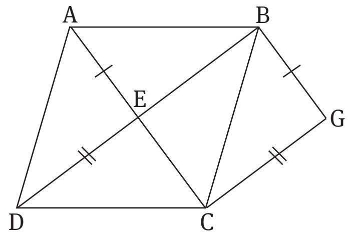
AE = CE	(given)
Then AD = DC	(all 4 sides equal)
DE = DE	(common side)
∴ ΔADE ≡ ΔCDE	(SSS)
∴ ΔADE = ΔCDE	

Questions

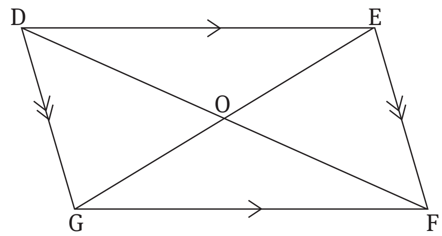
1 In the diagram, WX||ZY and ΔWXZ = ΔWYZ. Prove that WXYZ is a parallelogram.



2 In the diagram alongside, ABCD is a rhombus with BG = AE and CG = DE. Prove that EBGC is a rectangle.



3 Use the diagram alongside to prove that the diagonals of a parallelogram bisect each other.

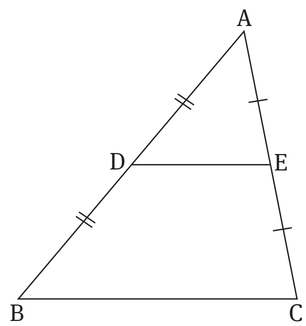


4 In the diagram alongside, AD = DB and AE = EC. Prove that:

a ΔADE|||ΔABC

b DE||BC

c DE = BC



Chapter 8

Analytical geometry

Overview

When answering questions on analytical geometry:

- ALWAYS make a sketch. It can be a rough sketch, but at least have the points in the correct quadrants of the Cartesian plane.
- Read carefully and put all the information in the sketch.
- Always ask yourself: Is this answer possible if I look at my drawing?
- When using a calculator, always round off at the final answer only.

Unit 1

The distance formula

The distance between the two points is given by the formula:

$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ (Remember, the answer is a length)

1.2 Using the distance formula

Example

1 Determine the distance between P(2; 5) and Q(−4; 1) on the Cartesian coordinate system.

$PQ^2 = (x_p - x_q)^2 + (y_p - y_q)^2$

$= (2 - (-4))^2 + (5 - 1)^2$ (Use the given values)

$= (6)^2 + (4)^2$

$= 36 + 16$

$= 52$

$\therefore PQ = \sqrt{52} = 7{,}21$ units (Usually rounded off to two decimal places)

2 The distance between A(−5; *k*) and B(7; −3) is 13 units. Determine the value(s) of *k*.

Here, point A can be anywhere along the vertical dashed line in the graph, as long as the distance between A and B is 8 units.

$AB^2 = (x_a - x_b)^2 + (y_a - y_b)^2$

Substitute in the values given:

$132 = (-5 - 7)^2 + (k - (-3))^2$

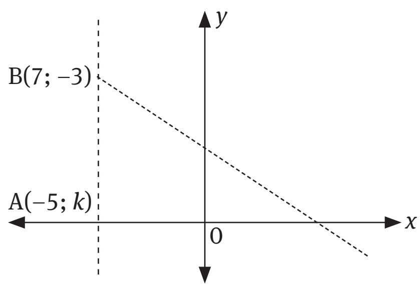
$169 = 144 + (k + 3)^2$

$= 144 + k^2 + 6k + 9$

$k^2 + 6k - 16 = 0$

$(k + 8)(k - 2) = 0$

$\therefore k = -8$ or 2



Unit 2

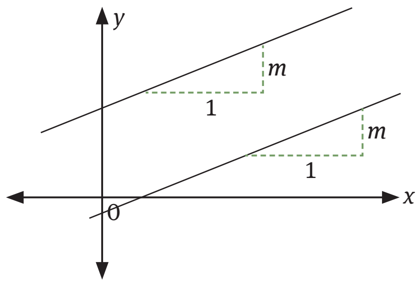
The gradient formula

The gradient of the line joining the points is given by the following formula:

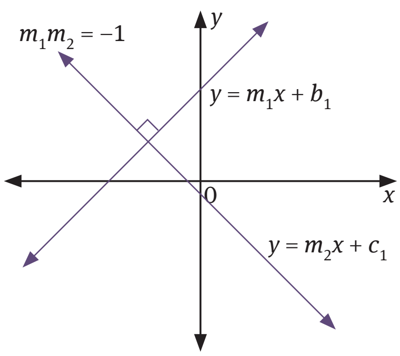
$m = \frac{y_1 - y_2}{x_1 - x_2}$

2.2 Parallel and perpendicular lines

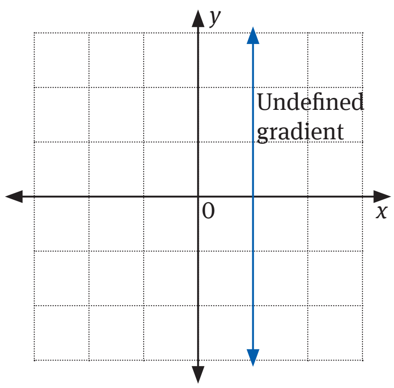
- When two lines are parallel, their gradients are the same.



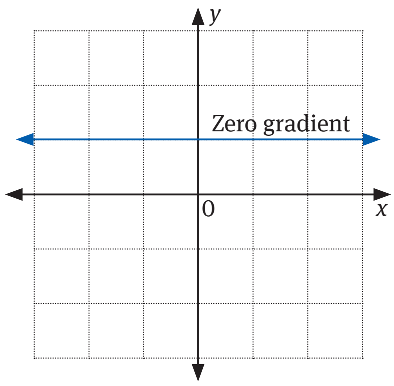
- When two lines are perpendicular then the **product** of their gradients equals −1.



- A vertical line has an **undefined** gradient.

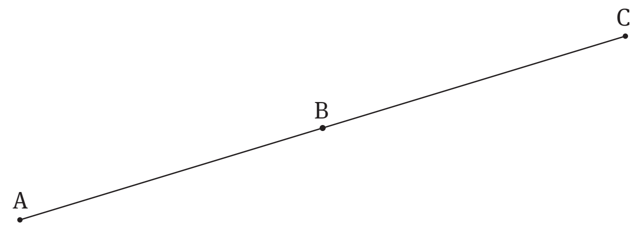


- A horizontal line has a gradient of 0.



2.3 Collinear points

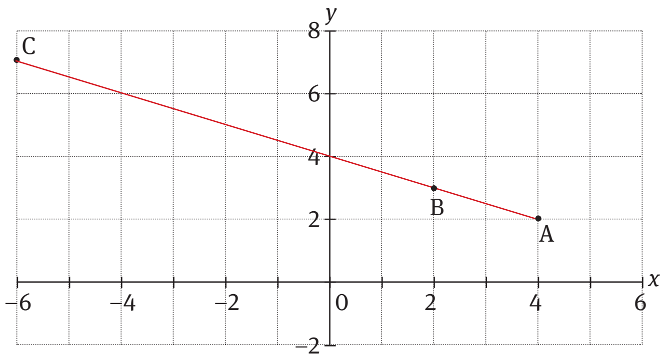
Points A, B and C are collinear if AB + BC = AC or if $m_{AB} = m_{BC}$ AND point B is a common point.



Example

1 Show that ABC is a straight line (in other words, show that A(4; –2), B(2; 3) and C(–6; 7) are collinear).

We need to calculate the gradients between AB and BC. If the gradients are the same, the points are collinear.



$$m_{AB} = \frac{y_A - y_B}{x_A - x_B} = \frac{2 - 3}{4 - 2} = -\frac{1}{2}$$

$$m_{BC} = \frac{y_C - y_B}{x_C - x_B} = \frac{7 - 3}{-6 - 2} = -\frac{1}{2}$$

Therefore, ABC is a straight line.

2 Determine the value(s) of k if P(8; –4); Q(k; 1) and R(–1; 3) are collinear.

If the points are collinear, then m_{PQ} must equal m_{PQ}. Therefore:

$$= \frac{y_p - y_q}{x_p - x_q} = \frac{y_r - y_q}{x_r - x_q}$$

$$= \frac{-4 - 1}{8 - k} = \frac{3 - 1}{-1 - k}$$ (Replace with coordinate values)

$-3(-1 - k) = 2(8 - k)$ (solve for k)

$3 + k = 16 - 2k$

$3k = 13$

$\therefore k = 4$

Unit 3

The midpoint formula

The coordinates of the midpoint of the line joining the points is given by:

$M\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$ (Remember, the answer is a set of coordinates)

Example

1 Determine the midpoint, M, of CD, if the points are C(–4: 5) and D(2; –3).

$$M\left(\frac{x_c + x_d}{2}; \frac{y_c + y_d}{2}\right) = M\left(\frac{-4 + 2}{2}; \frac{5 + (-3)}{2}\right)$$
$$= M(-1; 1)$$

2 Determine the coordinates of P if Q is (–1; 5) and point R(3; 7) is the midpoint of PQ. Suppose the coordinates for P are $(x_p; y_p)$. Use the midpoint formula to solve two equations, one for the *x*-value of P and one for the *y*-value for P.

$$(3; 7) = \left(\frac{x_p + x_q}{2}; \frac{y_p + y_q}{2}\right)$$

x-coordinate of the midpoint:

$$x_R = \frac{x_p + x_q}{2}$$
$$3 = \frac{x_p + (-1)}{2}$$
$$6 = x_p - 1$$
$$\therefore x_p = 7$$

y-coordinate of the midpoint:

$$y_R = \frac{y_p + y_q}{2}$$
$$7 = \frac{y_p + 5}{2}$$
$$14 = y_p + 5$$
$$\therefore y_p = 9$$

$\therefore$ P is the point (7; 9)

Questions

1 Given: P(–4; –1); Q(6; 3); R(6; b) and S(–4; –3):

a Determine the gradient of PQ.

b If PQ is parallel to SR, determine the value of b.

c Show that PQ = SR.

d Is quadrilateral PQRS a parallelogram? Give a reason for your answer.

e Calculate the midpoint of:

i PR ii SQ

f What can you subsequently deduce regarding the diagonals of a parallelogram?

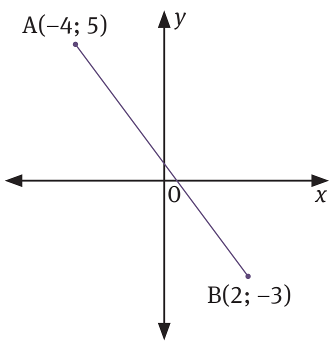
g A rhombus is a parallelogram with two consecutive sides that are equal in length. Show that PQRS is not a rhombus.

2 Using the sketch alongside (not drawn to scale), calculate:

a AB

b the midpoint, K, of AB

c the gradient of A



3 BD and AC are the diagonals of a parallelogram. If B = (2; 3), D = (6; 0), C = (7; 5) and A = (x; y), find the values of x and y.

4 Given points A(3; 7), B(5; 11) and C(6; 3), find:

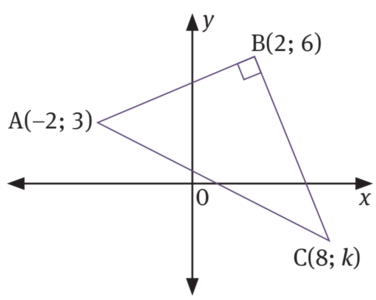
a the length of AB, leaving your answer as a surd.

b the length of BC, correct to two decimal places

c the midpoint of AC

d the gradient of AB and BC

5 Triangle ABC (alongside) is made up of the points A(–2; 3), B(2; 6) and C (8; k). Find the value of k, given that ∆ABC = 90°.



6 G(3; 7), H(–5; 1), K(1; –3) and D(x; y) are the vertices of a parallelogram.

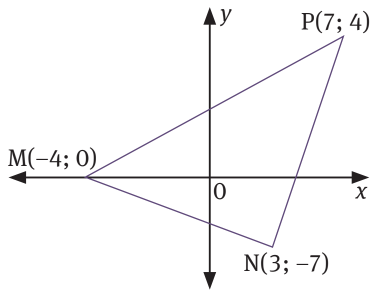
a Calculate the length of the line HK.

b Calculate the length of KD.

c Find the coordinates of D.

d Find the coordinates of the midpoint of HD.

7 The points M(–4 0), N(3; –7) and P(7; 4) are shown in the diagram alongside. Calculate the following:



a Calculate the midpoint, Q, of MN.

b Calculate the gradients of MN and PQ

c Calculate the product of the gradients of MN and PQ.

d What can you say about MN and PQ?

e Calculate the lengths of PM and PN.

f Draw a conclusion about ΔPMN.

Chapter 9

Finance, growth and decay

Overview

Interest is a fee paid to borrow money. It is usually charged or paid as a percentage of the total amount borrowed or invested. An interest rate is the cost stated as a percentage of the amount borrowed/invested per period of time, usually one year.

Simple interest and compound interest are examples of how interest can be calculated. The compound interest formula can also be used in other areas in which compound growth occurs, such as inflation, exchange rates and population growth.

CHAPTER 9 Page 91
Finance, growth and decay

UNIT 1 Page 92
Simple interest

- Calculating simple interest
- Hire purchase

UNIT 2 Page 94
Compound growth

- Compound interest
- Applications of compound growth

Unit 1

Simple interest

Simple interest is calculated as a constant percentage based only on the amount of money invested/borrowed. Here, we do not receive or pay interest on any interest accrued during the period of the loan or investment.

1.1 Calculating simple interest

- Simple interest is calculated on the original principal amount only.
- The formula is: simple interest = $P \times i \times n$, where
  - P = principal (original amount borrowed or loaned)
  - i = interest rate for one period
  - n = number of periods

Example

1 You borrow R10 000 for 3 years at 5% simple interest, paid annually.
P = R10 000; $n = 3$; $i = 0{,}05$ ($5\% = \frac{5}{100} = 0{,}05$)
interest = $P \times i \times n = 10\,000 \times 0{,}05 \times 3$ = R1 500

2 a You borrow R10 000 for 60 days at 5% simple interest per year (assume a 365 day year).
P = R10 000; $n = (60 \div 365)$; $i = 0{,}05$
interest = $P \times i \times n = 10\,000 \times 0{,}05 \times (60 \div 365)$ = R1 500
Note: interest is calculated per year, but the loan is only for 60 days. This is a fraction part of the year. Therefore, $n = (60 \div 365)$.

b After the 60 days, you have to pay the money back, with interest.
$A = P$ + simple interest
= R 10 000 + R82,19
= R10 082,19

$A = P + P \times i \times n$ (take out a common factor)
$A = P(1 + in)$

Example

An amount, b, is invested at 7,5% p.a. for 3 years to yield simple interest amounting to R500. Find the value of b.

$P \times i \times n = \text{Simple interest}$

$P = b;\ i = \frac{7,5}{100} = 0,075;\ n = 3$

$\frac{b \times 7,5 \times 3}{100} = 500$

$b = \text{R}2\ 222,22$

1.2 Hire purchase

Hire purchase is a system in which a buyer pays for something in regular instalments while enjoying the use of it. During the repayment period, the buyer does not own whatever was bought. The buyer only takes ownership once the loan has been paid in full.

Example

Jean wants to buy a motorbike for R10 500. He pays a deposit of R2 000. He wishes to pay the balance using a hire purchase agreement over 3 years. The interest charged on the loan is 18% per annum. Included in the agreement, is an insurance cost of 2% per annum on the purchase price of the motor bike. Calculate his monthly instalment.

Balance owing after deposit = R8 500
Insurance = (2% of R10 500) × 3 years = R630
Interest on R8 500 = $P \times i \times n$
= R8 500 × 18 × 3
= R4 590

Monthly Payment = $\frac{8\ 500 + 630 + 4\ 590}{36}$

= R381,11

Unit 2

Compound growth

2.1 Compound interest

- Compound interest is calculated each period on the original principal and all interest accumulated during past periods.
- Although the interest may be stated as a yearly rate, the compounding periods can be yearly, semi-annually, quarterly, or even continuously.
- You can think of compound interest as a series of back-to-back simple interest contracts. The interest earned in each period is added to the principal of the previous period to become the principal for the next period.

For example, you borrow R10 000 for three years at 5% annual interest compounded annually:

- interest year 1 = $P \times i \times n$ = 10 000 × 0,05 × 1 = 500
- interest year 2 = $(P_2 = P_1 + i_1) \times i \times n$ = (10 000 + 500) × 0,05 × 1 = 525
- interest year 3 = $(P_3 = P_2 + i_2) \times i \times n$ = (10 500 + 525) × 0,05 × 1 = 551,25
- Total interest earned over the 3 years = 500 + 525 + 551,25 = 1 576,25

Compare this to 1 500 paid over the same period using simple interest.

The formula for compound interest is $A = P(1 + i)^n$, where

- A = the amount at the end of a loan/investment period
- P = the principle (initial amount borrowed/invested)
- i = the interest rate (expressed as a decimal number)
- n = the period of the loan/investment

Example

An amount of R1 500,00 is deposited in a bank paying an annual interest rate of 4,3%, compounded *quarterly*. What is the balance after 6 years?

Using the compound interest formula, we have:
$P = 1\ 500$, $i = 0{,}043$, $n = 6$

However, because the interest is calculated quarterly, we divide i by 4 and multiply n by 4. Therefore:

$$A = 1\ 500\left(1 + \frac{0{,}043}{4}\right)^{4(6)} \approx R1\ 938{,}84$$

So the balance after 6 years is approximately R1 938,84

2.2 Applications of compound growth

2.2.1 Inflation

Inflation refers to an ongoing general increase in prices.

Use the compound growth formula if you need to work out how much something will cost in future: $A = P(1 + i)^n$

Inflation means that we can buy less in future, so if you need to work out what is something worth in future, just an adjusted formula: $A = P(1 - i)^n$

Example

If the average rate of inflation for the past few years was 7,3%, and your family's water and electricity bill is on average R1 425, what can you expect to pay in 6 years' time?
$A = P(1 + i)^n$

2.2.2 Exchange rates

- An exchange rate gives the relationship between two countries' currencies.

Example

1 Ruff wants to import a TV from England at a price of £507, plus a delivery cost of 15% and import duty of 20%. The equivalent TV locally costs R13 000. The exchange rate is R14,08 to the pound.
Show, with necessary calculations, whether Ruff should import or buy locally.
£507 × 14,08 = R7 138,56
Delivery cost = R7 138,56 × 15% = R1 070,78
Import duty = R7 138,56 × 20% = R1 427,71
Total cost = R 9 637,05
Conclusion: It costs less to import the TV

2 If the exchange rate is 1 euro = R8,1671 and 1 pound sterling = R12,1668, determine the exchange rate between the euro and the pound.
1 euro = R8,1671
1 rand = 1 ÷ 8,1671 = 0,1224 euro
1 pound sterling = R12,1668
1 rand = £(1 ÷ 12,1668) = £0,0822
0,1224 euro = £0,0822
1 euro = £0,6713

2.2.3 Population growth

- Since we are referring to growth, we use the formula $A = P(1 + i)^n$.
- Here, i = rate of population growth and P = initial size of the population.

Example

There are 12 500 people in a small town. The population of the town increases every year by 5,5%. What will the population of the town be after 5 years?

$A = 12\ 500(1 + 0{,}055)^5$

$A \approx 16\ 337$

Questions

1 Mr du Toit decided to invest an amount of money at an interest rate of 9% p.a. compounded annually. He would like R70 000 after six years. How much money must he invest now? (4)

2 A person invests R30 000 in a savings account for eight years at a simple interest rate of 6% p.a. Calculate the accumulated amount. (4)

3 A motor vehicle is purchased for R105 000. Of this, 20% is paid in cash and the balance is paid using a hire purchase agreement for seven years by means of equal monthly payments. The bank decides to charge an interest rate of 14,5% p.a. for the seven-year period. Calculate:

a The loan amount (4)

b The full amount including interest (2)

c The monthly repayments (3)

d The total amount paid if the insurance on the car per month is R780 (4)

4 A mother decides to buy a baby's pram from Kuwait. The price of the praxm is 80 Kuwaiti dinars. Calculate how much she will pay if the exchange rate is 1 Kuwaiti dinar to R26,09. (4)

5 A tourist from the United Kingdom buys computers from South Africa at a cost of R530 000. He wants to pay for these computers in British pounds. If the exchange rate is 1£ = R11,72, how much will he pay in pounds? (To the nearest pound) (4)

6 Mr Dube wants to buy a car. So he borrows R420 000 and agrees to settle this amount in eight years together with simple interest charged at 12,5% p.a. How much will he have to pay in eight years from now? (4)

7 Ronnie wants to receive an amount of R180 000 after 10 years. How much must he invest now at an interest rate of 7,25% p.a. compounded annually (to the nearest rand)? (4)

8 The population of Phoenix is growing at a rate of 4% p.a. compounded annually. If the population is now 50 000, what will be the percentage growth in five years time? (Give your answer correct to two decimal places) (4)

9 Oil costs $76 a barrel. The exchange rate is 1$ is equal to R7,23. If South Africa were to import 40 000 barrels , how much will the oil cost (in rand)? (4)

10 Matthew invests R30 000. The institution offers Matthew two investment options:

Option 1: 8 years at 12,5% p.a. simple interest.

Option 2: 8 years at 12,5% p.a. compound interest.

Which option should Matthew choose? Motivate your answer. (9)

Chapter 10

Statistics

Overview

Statistics is the branch of Mathematics that is used daily over a broad spectrum of common everyday life to gather and interpret information. This information or data enables us to predict the future, to make choices and to improve existing frameworks.

CHAPTER 10 Page 98
Statistics

UNIT 1 Page 99
Measures of central tendency: ungrouped data
- The mean, median and mode

UNIT 2 Page 101
Measures of dispersion
- The range
- Quartiles
- Interquartile range
- Semi-interquartile range
- Percentiles
- The five number summary

UNIT 3 Page 103
Box-and-whisker plot

UNIT 4 Page 104
Grouped data
- Measures of central tendency
- More about organising data

Unit 1

Measures of central tendency: ungrouped data

There are mainly two different types of numerical which could be collected, namely discrete and continuous data.

- Discrete data is data with specific values normally gathered by means of counting, e.g. we are three children in our family.
- Continuous data is data that can occupy any value between two points and is usually obtained through measurement, e.g. I am 1,76 m tall.

1.1 The mean, median and mode

Mean

The arithmetic mean is commonly known as the average. It is calculated by the sum of the values in the sample group ($\sum x$), divided by the number of values in the sample group (n).

Mean = $\frac{\sum x}{n}$

Median

The median is the middle value in the sample group, after they have been arranged in ascending order. In other words, it is the value in the middle of the ordered data. If there is an even number of values, the median is the average of the two middle values.

Mode

The mode is the value that appears most often. In other words, it is the value with the highest frequency.

How do we decide which measure to use? It is important to use the most appropriate measure of central tendency in order to make a sound conclusion. (You sometimes have to answer such a question in a test or exam.)

Measure	Advantages	Disadvantages
Average/Mean	• Most common measure • Used widely in media • Easy to calculate • All values in the sample group are used	• If an extreme (very large or very small) value is added, the average changes drastically • Cannot be used when the data value is simply a category
Median	• Easy to identify if the sample group is small	• Time consuming if the sample if very big and not in ascending order
Mode	• No calculation needed • Easy to deduce in grouped data • Is used to describe any kind of data	• A small sample usually does not have a mode • More than one mode is possible

Example

How long is the average name?
The number of letters in the first names of 26 learners was recorded as follows:

4	6	3	4	7	6	5	6	8	4	3	9	4
5	4	6	7	8	8	4	4	4	6	6	5	9

Arithmetic mean/average:

Mean $= \frac{\text{sum of all the letters}}{\text{number of names}} = \frac{145}{26} = 5,57... \approx 6$ letters

Median: First arrange in ascending order:

3	3	4	4	4	4	4	4	4	4	5	5	5
6	6	6	6	6	6	7	7	8	8	8	9	9

There is an even number of values, therefore:

Median $= \frac{\text{sum of the middle terms}}{2} = \frac{5 + 6}{2} = 5,5 \approx 6$ letters

Mode: 4 occurs most often and in thus the mode.

Unit 2

Measures of dispersion

2.1 The range

The range = largest value – smallest value.
It gives us an indication of how big the span of the data is. For example, does the data lie between 1 and 10 or 1 and 100?

2.2 Quartiles

The quartiles divide the data (in ascending order) into four quarters. Every quarter has the same number of items. There are three quartiles:

- The lower quartile (Q1)
- The middle quartile or median (Q2)
- The upper quartile (Q3)

2.3 Interquartile range

The interquartile range is the difference between the third quartile and the first quartile. It measures the range of the middle 50% of the data (in ascending order). A large interquartile range indicates the data values are widely dispersed.

- Interquartile range = Third quartile – First quartile
- $IQR = Q_3 - Q_1$

2.4 Semi-interquartile range

The semi-interquartile range is half of the range of the middle half of the data. It is seldom influenced by extreme values in a set of data, and so is a good measure of dispersion.

$$\text{Semi-interquartile range} = \frac{\text{Third quartile} - \text{First quartile}}{2}$$

2.5 Percentiles

- The percentiles divide the data into 100 equal parts.
- Every part has the same number of items.
- Percentiles give an indication of how many values in the data set are smaller than a certain value.
- For example, the 56th percentile of 70 means that 56% of the other values are less than 70.

2.6 The five-number summary

The five-number summary is five-point scale that we can use to summarise the information about a data set. The five-number summary contains:

- Minimum value
- First or lower quartile: 25% of the data lies below the first quartile
- Median: 50% of the data lies below and above the median
- Third or upper quartile: 25% of the data lies above the third quartile
- Maximum value

Example

Consider the following values

12	6	4	9	8	4	9	8	5	9	8	10	7	
9	12	14	5	8	4	6	6	4	8	2	6	5	10

The first thing to do when confronted with a new set of data is to arrange it in ascending order:

2	4	4	4	4	5	5	5	6	6	6	6	7	
8	8	8	8	8	9	9	9	9	10	10	12	12	14

- Range = 14 – 2 = 12
- Minimum value = 2
- First quartile (Q_1) = T7 = 5 (For discrete data: 0,25 × 27 = 6,75 thus 7th term)
- Median = 8
- Third quartile(Q_3) = T21 = 9 (For discrete data: 0,75 × 26 = 20,25 thus 21st term)
- Maximum value = 14
- Inter-quartile range = 9 – 5 = 4
- Semi-inter-quartile range = $\frac{\text{Third quartile} - \text{First quartile}}{2} = 2$

Unit 3

Box-and-whisker plot

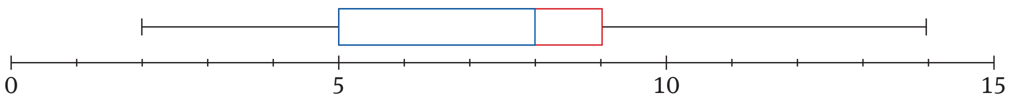
The box-and-whisker plot is a graphical representation of the five-number summary.

- The left side of the box is the lower quartile
- The vertical line inside the box is the median
- The right side of the box is the upper quartile.
- The two lines on either side of the box extend out to the minimum and maximum values.

Example

Below is the box-and-whisker plot for our earlier earlier example:

- Minimum value = 2
- First quartile (Q_1) = T7 = 5 (For discrete data: 0,25 × 27 = 6,75 thus 7th term)
- Median = 8
- Third quartile (Q_3) = T21 = 9 (For discrete data: 0,75 × 26 = 20,25 thus 21st term)
- Maximum value = 14



Unit 4

Grouped data

4.1 Measures of central tendency

A frequency table is used to sort and count information.

Earlier, we collected data about the first names of 26 learners:

4 6 3 4 7 6 5 6 8 4 3 9 4

5 4 6 7 8 8 4 4 4 6 6 5 9

We can count how many times each length of name occurs and summarise this information in a frequency table. In this way, we group the data into categories, or class intervals.

Total letters	Tally	Frequency
3	||	2
4	~~||||~~ |||	8
5	|||	3
6	~~||||~~ |	6
7	||	2
8	|||	3
9	||	2

The maths marks of 220 grade 10 learners at a school can be summarised as follows:

Percentage (Class interval)	Number of learners (frequency)
	4
	10
	37
	43
	36
	26
	24
	20

$\sum f = 200$

How to calculate the approximate mean of grouped data:

- Step 1: Determine the midpoint for each interval.
- Step 2: Multiply the class midpoint by the frequency.
- Step 3: Add up the results from Step 2.
- Step 4: Divide the total from Step 3 by the frequency.

Class intervals	Frequency (f)	(Step 1) Midpoints (x)	(Step 2) $f \times x$
	4	9,5	38
	10	24,5	245
	37	34,5	1 276,5
	43	44,5	1 913,5
	36	54,5	1 962
	26	64,5	1 677
	24	74,5	1 788
	20	84,5	1 690
	0	94,5	0
	$\sum f = 200$		$\sum fx = 10\,590$ **(Step 3)**

Step 4: Approximate mean = $\sum fx \div \sum f = 10\,590 \div 200 = 52{,}95$

Central tendency	In which class it falls	
Median	$50 \leq x < 60$	(The middle (100th) score is in this class)
Lower quartile	$30 \leq x < 40$	(25% or less of the marks are in this class)
Upper quartile	$60 \leq x < 70$	(More than 75% of the marks are in this class or lower)
13th percentile	$30 \leq x < 40$	(13% is about 26 marks and lower – the frequency up to this class is 51)
Modal class	$40 \leq x < 50$	(The class with the highest frequency)

- The midpoint of a class: $\frac{\text{highest value of the interval} - \text{lowest value}}{2}$
- $\sum f$: Sum of the frequencies
- The position of median: $\frac{n+1}{2}$

More about organising data

The following table shows the costs (in rand) of 50 new members of a cellphone company. The company would like touse the data to help with its planning.

42,19	5,64	83,26	99,56	3,43
38,45	91,1	27,56	99,5	106,84
95,73	16,44	3,69	2,42	115,78
104,8	99,03	5,1	23,41	21
22,57	14,34	18,49	21,13	72,02
92,97	20,55	15,3	6,48	15,42
88,62	117,69	29,23	10,88	64,78
115,5	33,69	103,15	109,08	19,34
119,63	92,17	74,01	29,24	3,03
13,26	11,27	63,7	79,52	85,67

When large amounts of data need to be studied and interpreted, it is sometimes necessary to group the data, and then study the each group.

It is important to decide how many groups you would like to use before you start organising the data. In this case, we will use six groups for the data.

The size of each group is: largest value – smallest value divided by the number of groups.

Group size: $\frac{\text{Largest number} - \text{smallest number}}{\text{number of groups}} = \frac{119{,}63 - 2{,}42}{6} = 19{,}5 \approx 20$

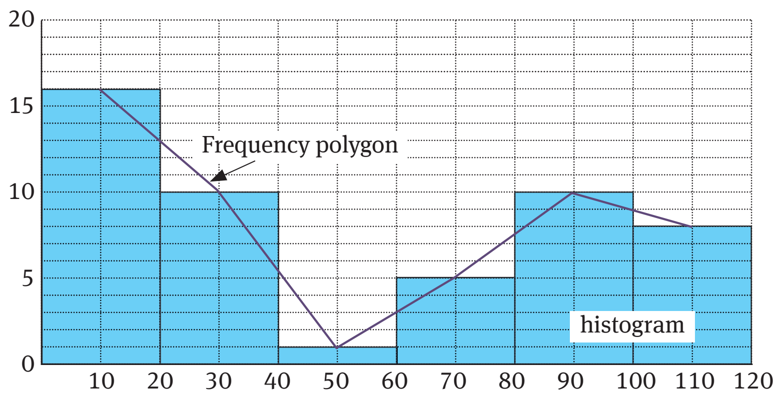
The next step is to draw up a frequency table.

Telephone costs (R)	Frequency (f)	Midpoint (x)	$f \times x$
0 – 19	16	9,5	152
20 – 39	10	29,5	295
40 – 59	1	4,5	49,5
60 – 79	5	69,5	347,5
80 – 99	10	89,5	895
100 – 120	8	109,5	876
	$n = 50$		2 615

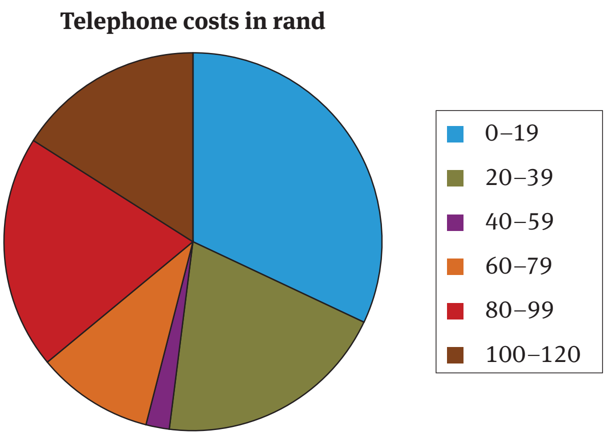
Estimated average $= \frac{\Sigma f \times x}{50} = \frac{2\,615}{50} = 52{,}3$

Arithmetic mean $= \frac{\Sigma x}{50} = \frac{2\,647{,}7}{50} = 53$

We can represent this data graphically in a histogram or a frequency polygon.

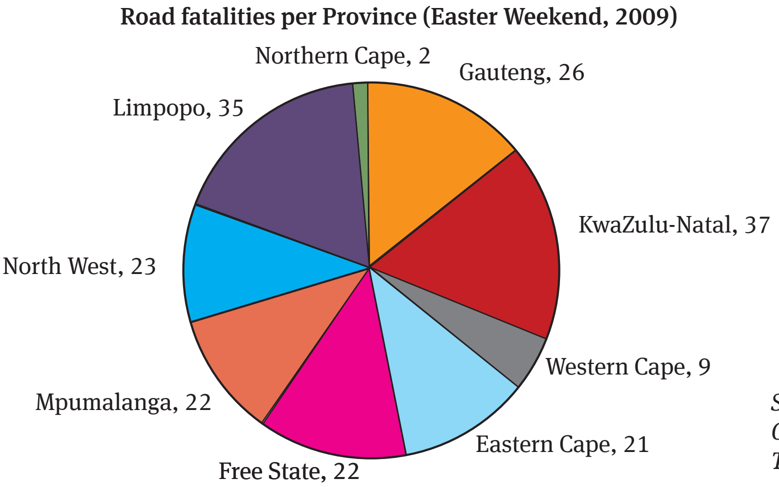


We can also represent the data in a pie chart, as follows.



Questions

1 Consider the pie chart below showing the number of deaths on South African roads over the Easter weekend of 2009.



Source: Road Traffic Management Corporation, Easter 2009 Road Traffic Report; Wed, 15 April 2009

a How many people lost their lives due to road accidents in South Africa during the Easter weekend of 2009?

b Which province had the highest number of road fatalities?

c Why does Northern Cape such a small number of road fatalities?

d What percentage of road deaths occurred in KwaZulu-Natal?

2 The following table shows the heights (in cm) of the learners in a class.

163	137	146	166	163	134	146	141	134	157
147	141	134	157	136	158	157	188	158	144
157	145	144	158	144	146	137	134	137	188

a Determine the range, mean, median and mode of the heights in the class.

b Based on the range, do you think that the heights of the learners vary considerably, or do the learners have fairly similar heights? Explain your answer.

3 The following table shows the percentage test marks for the learners in a class. The marks have been organised separately for the female and male learners.

Females								
70%	52%	48%	64%	54%	66%	50%	68%	62%
52%	70%	60%	66%	56%	58%	56%	70%	
Males								
52%	74%	60%	22%	60%	24%	90%	58%	62%
70%	46%	74%	40%	34%				

a Determine the range for the marks of the female learners.
b Determine the range for the marks of the male learners.
c Compare the range values for the female and male learners. What does the difference between these values tell you about the spread of the marks for each of these groups.
d i Complete the following frequency table to organise the marks for the female and male learners.

	Females	Males
Mark category	No. of learners who achieved marks in this category	No. of learners who achieved marks in this category
0 – 29%		
30 – 39%		
40 – 49%		
50 – 59%		
60 – 69%		
70 – 79%		
80 – 89%		
90 – 100%		

ii Draw two separate histograms to represent the marks scored by the females and the males. You will need to construct your own axes for the graphs.
iii Compare the two graphs and describe what you notice about the difference between the spread of the marks for the females and the males.
iv Do you think the female learners or the male learners performed better in the test? Explain your answer.

4. Three learners are applying for a bursary and have to submit marks for all their subjects in their application. The marks are given below:

Learner A: 55% ; 33% ; 22% ; 69% ; 58% ; 29%

Learner B: 44% ; 11% ; 33% ; 22% ; 55% ; 49%

Learner C: 53% ; 54% ; 64% ; 49% ; 45% ; 64%

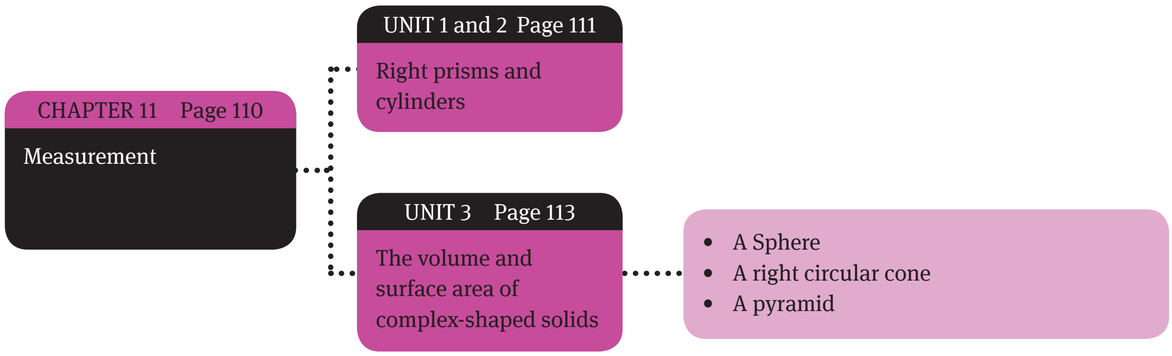
a Determine the average (mean) of the marks for each learner.
b Determine the range of the marks for each learner.
c If you were responsible for deciding which student should get the bursary, who would you choose? Give reasons for your answer using the mean and the range values.

Chapter 11

Measurement

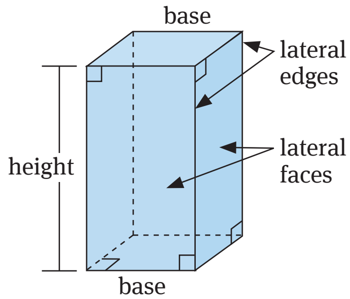
Overview

In this chapter, we focus on three-dimensional (3D) solids. The surface area of a 3D solid refers to the area of the outside surface of the solid. The volume refers to the amount of space inside the 3D solid.

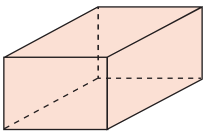
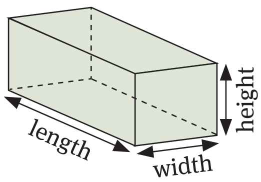
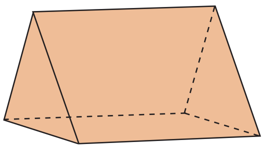
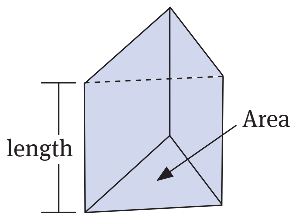
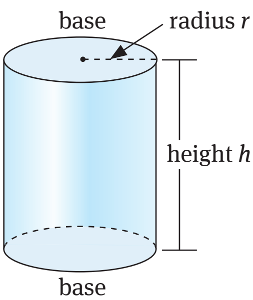
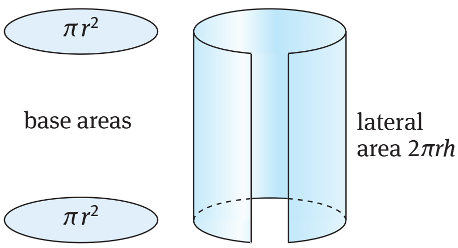
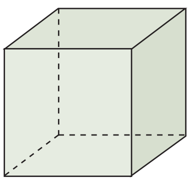
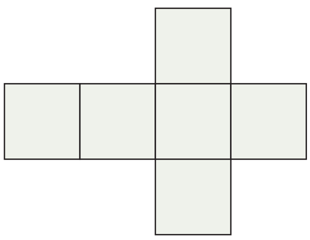


Right prisms and cylinders

- A prism is a polyhedron (solid) with two congruent faces, called bases, that lie in parallel planes.
- The other faces, called lateral faces, are parallelograms formed by connecting the corresponding vertices of the bases.
- The segments connecting these vertices are lateral edges.
- A prism can be cut into slices, which are all the same shape.

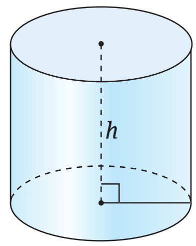


Right rectangular prism

Prism	Volume	Surface area
Rectangular prism 	length × breadth × height	$2lw + 2lh + 2wh$ 
Triangular prism 	$\frac{\text{base} \times \text{height}}{2} \times$ height of prism	$b.h + (S_1 + S_2 + S_3) \times h$ 
Cylinder 	$V = \pi r^2 \times h$	$2\pi rh + 2\pi r^2$ 
Cube 	$V = s^3$ s = side length of cube	Area of six cubes: Surface area = $6s^2$ 

Example

Find the height of a cylinder which has a radius of 6,5 cm and a surface area of 592,19 cm^2.



1 Make h the subject of the formula.
2 Substitute values into the formula.
3 Do not leave answer in terms of π, unless specifically asked to do so.
4 Do not forget the units.

$$\text{Surface area} = 2\pi rh + 2\pi r^2$$
$$2\pi rh = \text{SA} - 2\pi r^2$$
$$h = \frac{\text{SA} - 2\pi r^2}{2\pi r}$$
$$h = 592{,}19 - 2\pi(6{,}5)^2$$
$$h \approx 8 \text{ cm}$$

Example

A right circular prism (cylinder) has a volume of 50 $units^3$, a radius r and height h. If the radius is tripled and the height halved, what is the new volume of the cylinder?

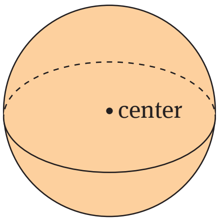
$$V_{new} = \pi(3r)^2\left(\frac{h}{2}\right)$$
$$= 9r^2\pi\left(\frac{h}{2}\right)$$
$$= \left(\frac{9}{2}\right)\pi r^2 h \qquad (\pi r^2 h = 50)$$
$$= \left(\frac{9}{2}\right) \times 50$$
$$= 225 \text{ units}^3$$

Unit 3

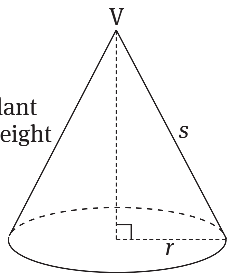
The volume and surface area of complex-shaped solids

3.1 A sphere

- Surface area = $4\pi r^2$
- $V = \frac{4}{3}\pi r^3$

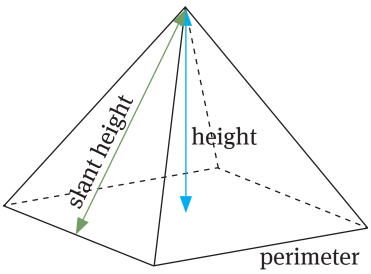


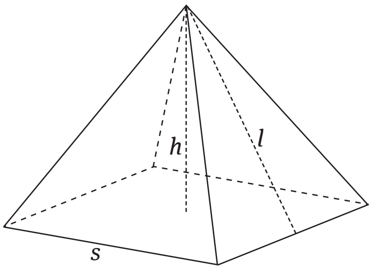
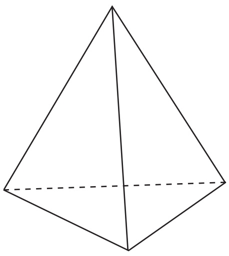
3.2 A right circular cone

A cone is simply a pyramid with a circular base.  Circumference of the base = $2\pi r$	$V = \frac{1}{3}\pi r^2 h$	$\pi rs + \pi rs^2$ • Area of the cone section + area of circle • Curved surface area (without the base) = πrs • s = slant height

3.3 A pyramid

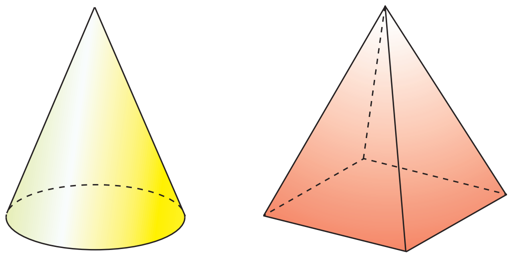
A pyramid is made by connecting a base to an apex.



Rectangular base pyramid 	$V = \frac{1}{3}Ah$ A = Area of the base	Area of the base + $\frac{1}{2}$ perimeter of base × slant height
Triangular base pyramid (also called a tetrahedron) 	$V = \frac{1}{3}Ah$ A = Area of the base $V = \frac{1}{3}Ah$ $V = \frac{1}{3} \times \frac{1}{2} \times b \times h_{base} \times H_{pyramid}$ $V = \frac{1}{6}bh \times H$	4 × Area triangle: Area of regular triangle $= \frac{\sqrt{3}}{4} \times side^2$ $SA = 4 \times \frac{\sqrt{3}}{4} \times side^2$ $\therefore SA = \sqrt{3} \times side^2$

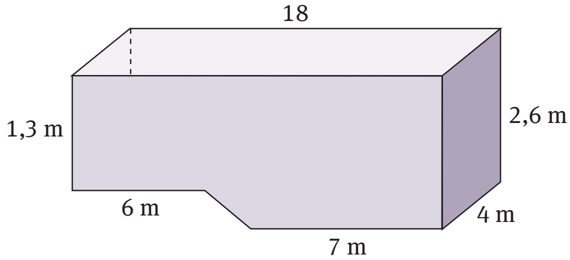
Example

1 Consider two types of containers, each 15 cm deep. One is a rectangle pyramid, with a base of 4 cm by 7 cm, and the other is a cone with radius 3 cm. Determine which container holds more water when full, and by how much?



Volume of pyramid $= \frac{1}{3} \times 4 \times 7 \times 15 = 140 \text{ cm}^3$
Volume of cone $= \frac{1}{3} \times 3 \times 3 \times 15 \times \pi = 45\pi \text{ cm}^3$
Difference $= 45\pi \text{ cm}^3 - 140 \text{ cm}^3 = 1{,}37 \text{ cm}^3$
Therefore, the cone holds more water.

2 The sketch alongside shows the cross-section of a swimming pool. Determine the surface area and the volume of the pool. (Remember: the top is open!).



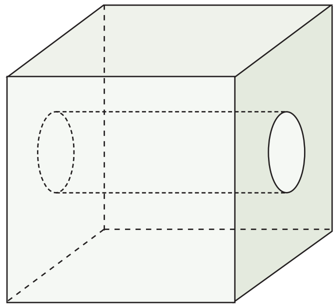
Surface area $= (2{,}6 \times 4) + (1{,}3 \times 4) + (6 \times 4) + (7 \times 4) + (4 \times \sqrt{26{,}69}) + 2(1{,}3 \times 18)$
$+ 2 \times \frac{1{,}3}{2}(12 + 7)$
$= 159{,}76 \text{ m}^2$
Volume $= [(1{,}3 \times 18) + \frac{1{,}3}{2}(12 + 7)] \times 4$
$= 143 \text{ m}^3$

Note: If you have a cone with no base, don't add the base area. There are different ways to calculate the surface area of a cone. Remember the basic concept: you add the cone's slanted area to the cone's base area.

Questions

1 A cube with a side length of 12 cm is filled to the top with water. The water is carefully poured into a rectangular prism with a length of 18 cm and a width of 8 cm. Calculate the height of the water in the rectangular prism.

2 A cylindrical hole has been drilled through the centre of a 10 cm solid cube (see figure alongside). The diameter of the cylindrical hole is 3 cm and its height is perpendicular to the two opposite faces of the cube. What is the total surface area of the cube (correct to two decimal places)?



3 A hollow sphere (e.g. a tennis ball) has an interior radius of 15 mm and an exterior radius of 20 mm. Calculate the volume of the material forming the sphere in cubic centimetres.

4 A metal top in the shape of a cone has a perpendicular height of 70 mm. If it displaces 4 224 cm^3 of water when fully immersed, calculate the total surface area.

Chapter 12

Probability

Overview

Probability refers to the likelihood or chance of an event taking place. Probability is used on a daily basis by businesses, insurance companies, engineers, climatologists and even doctors to predict future events.

Unit 1
Probability

- Probability is the study of chance.
- Probability is calculated as follows: $P(A) = \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$

1.1 The language of probability

To illustrate each of the following terms, consider rolling a normal six-sided die. Suppose we want to know the probability of rolling an odd number:

- Outcome: the result of an experiment (roll an odd number)
- Sample space: the set of all possible outcomes (set {1; 2; 3; 4; 5; 6})
- Event: a subset of the sample space (set {1; 3; 5})
- An event that will definitely take place has a probability of 1.
- An event that can never take place has a probability of 0.
- All other events fall somewhere between these two extremes.
- Probability can be expressed as a fraction, decimal fraction or percentage.

1.2 Theoretical probability

The outcome of rolling a die will be a number from 1 to 6. We want to know the probability of a successful outcome, which is one that matches an event. If we let $n(S)$ be the number of items in the sample space, $n(A)$ be the number of items in event A, and P(A) be the probability that A will occur:

$$P(A) = \frac{\text{number of successful outcomes}}{\text{total number of all possible outcomes}} = \frac{n(A)}{n(S)}$$

In our example, A is the possibility of rolling an odd number. Since there are three odd numbers between 1 and 6, we have:

$$P(\text{rolling an odd number}) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

1.3 Relative frequency

The relative frequency is the actual frequency of an event during an experiment. In practice, many instances occur where the probability of an event is not easy to calculate. The only way to determine the probability of such an event is to determine its relative frequency through many experiments.

- Relative frequency $= \frac{x}{N}$ (where: x = total favourable outcomes and N = total possible outcome in the sample space.)

Example

Cat and his friend Ratatouille learnt in their Maths class that the probability of a coin falling on heads is 50%, but struggle to believe it. Cat is the captain of the U16 soccer team and has lost many a coin toss on a Saturday morning. The two friends decide to test the theory experimentally. Cat flipped the coin and Ratatouille recorded the results.

The results after 80 flips were as follows:

H	H	T	H	T	T	H	H	H	T	H	T	T	T	H
T	H	T	H	T	T	H	T	T	H	H	T	T	T	H
H	T	T	T	H	H	H	H	T	T	T	T	H	H	T
H	T	H	T	H	T	T	T	H	T	H	T	T	T	H
T	H	T	H	T	H	H	T	H	T	H	H	T	T	H
H	H	H	H	H										

According to the data, what is the relative frequency of heads (H) after:

1 10 throws? 2 20 throws? 3 30 throws? 4 80 throws?

1 After 10 throws: RF = $\frac{6}{10}$ = 0,6

2 After 20 throws: RF = $\frac{10}{20}$ = 0,5

3 After 30 throws: RF = $\frac{14}{30}$ = 0,47

4 After 80 throws: RF = $\frac{40}{80}$ = 0,5

The relative frequency of an event is predicted with the aid of an experiment or an investigation.

Unit 2

Combination of events

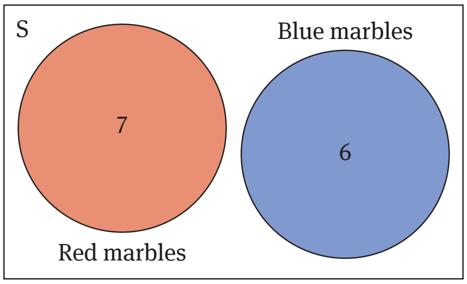
2.1 Venn diagrams

Venn diagrams are used to visually represent the sample space of an experiment and the relationship between the events in the sample space.

Remember: The sample space contains all the possible outcomes of an event.

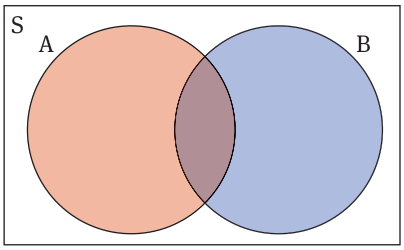
For example, suppose a container contains 7 red marbles and 6 blue marbles. The collection of red marbles is a subset of all the marbles. The same is true for the collection of blue marbles.

- Number of marbles in the sample space $= n(S) = 13$
- Number of marbles in the red subset$= n(R) = 7$
- Number of marbles in the blue subset $= n(B) = 6$
- thus: $n(S) = n(R) + n(B)$



2.2 Using Venn diagrams to solve probability problems

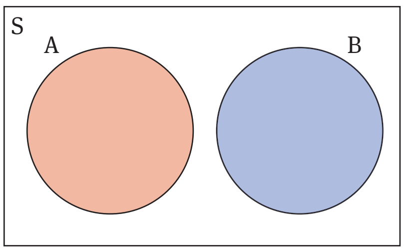
- The union of two collections, A and B, contains all elements that occur in A **or** B: Union of A or B = $A \cup B$
- The intersection of two collections, A and B, contains all elements that occurs in both A **and** B: Intersection of A and B = $A \cap B$
- Two collections are **disjoint** if there are no elements belonging to both collections. In other words: $n(A \cap B) = 0$.



Mutually exclusive events

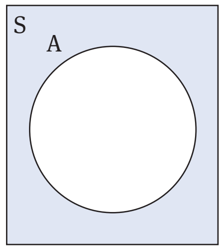
In this case, if A happens, B cannot happen, and vice versa.

- P(A OR B) = P(A) + P(B)
- The events cannot both happen at the same time.
- The sets of outcomes for the events are disjoint (have no elements in common). Therefore P(A B) = 0
- $P(A \cup B) = \frac{n(A) + n(B)}{n(S)} = P(A) + P(B)$
- $n(A \cup B) = n(A) + n(B)$



Complementary events

Events A and B are complementary if, together, A and B form the complete sample space.



- $n(\text{A}) + n(\text{B}) = n(\text{S})$
- P(A) + P(B) = 1 P(A) = 1 – P(B)
- A = B′ = complement of A
- If the events A and B are not mutually exclusive, then:
  P(A or B) = P(A) + P(B) – P(A and B)

Independent events

The outcomes of independent events do not influence each other. When A and B are independent events: P(A AND B) = P(A) × P(B).

Questions

1 Forty-four learners plan to go camping over the long weekend. Two of the learners say that they have too many tests to write the next week and are going to stay at home. The rest then decide to camp in the Cederberg mountains. They have to decide if they want to camp in tents (T) or hire bungalows (R). Of the learners, 28 voted for camping in tents and 17 for bungalows (Note: some learners voted for both).

a Draw a Venn diagram to represent the information.

b Use the Venn diagram to determine the following:

i $n(T \cap R)$ ii P(T)

iii $P(T \cup R)$ iv $n(R')$

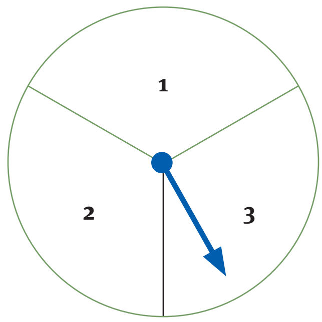
v $n(S)$ vi n(T or R)

2 Suppose you flip a coin and flick the pointer in the diagram alongside.

a How many outcomes are possible? List these.

b Determine P(heads and the value of 2).

c Determine P(tails and an uneven number).



Questions

Mid-Year Exam Paper

Question 1

a Give a value for x that will make $\frac{5x + 4}{\sqrt{x} - 4x}$

  i imaginary ii undefined (2)

b Consider the following three rows containing true statements:

  Row 1: $3^2 - 2 = 3 + 2^2$

  Row 2: $4^2 - 3 = 4 + 3^2$

  Row 3: $5^2 - 4 = 5 + 4^2$

  i Find two other rows of numbers for which similar statements are true.

  ii Describe a conjecture about these rows of numbers.

  iii Use a variable to write an algebraic statement for your conjecture to generalise the relationship between the left-hand side and the right-hand side.

  iv Hence prove or disprove your conjecture.

  v Write down the nth row of numbers. (11)

[13]

Question 2

a Simplify:

  i $\frac{2^{n-2}.3^{n+3}}{6^n}$ (4)

  ii $\frac{4x + 12}{3} - \frac{3x + 16}{4}$ (4)

b Multiply and simplify: $(2x + 3)(3x^2 + 2x - 1)$ (4)

c Factorise:

  i $2x^2 + 5x - 12$ (2)

  ii $12an - 8a + 3mn - 2m$ (3)

  iii $x^2(x + 2) - 4(x + 2)$ (4)

[21]

Question 3

a $10 - 2a \leq 84$ (3)

b $5.2^{7-2a} = 40$ (3)

[6]

Question 4

Solve for x and y: $3x + 2y = 5$ and $2x - 3y = 12$ [6]

Question 5

Solve for x:

a $(x - 2)(x + 3) = 6$ (4)

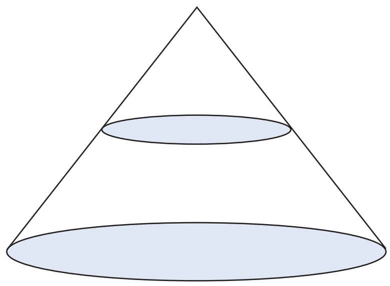
b $\frac{x^2 - 4x + 3}{3 - x} = \frac{x + 1}{2}$ (6)

c $x(x + 2) = 5x(x + 2)$ (2)

[12]

Question 6

A conical tank is shown alongside. The height of water at any point is $h = 6 - \frac{3}{4}r$, where r is the radius of the water surface in metres.



a Find the depth of the full tank.

b i How does the height (h) change with every 1 m decrease in r?

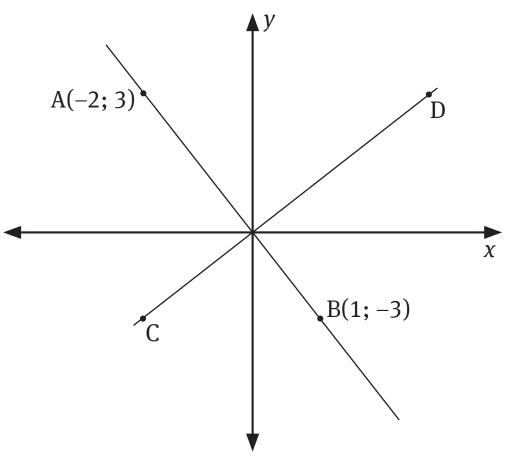
ii Find the radius of the bottom of the tank.

iii What is the circumference of the bottom of the tank? Give your answer correct to two decimal places.

iv How much water does the tank hold when full? [10]

Question 7

In the following sketch, the lines AB and CD intersect on the x-axis, with AB⊥CD.



Calculate:

a the length of the line AB leaving your answer in surd form. (3)

b the midpoint of AB (2)

c the equation of AB (4)

d the equation of CD (4)

[13]

Question 8

Consider the following raw data: 2; 4; 7; 11; 11; 15; 18; 19; 22; 31

Calculate and write down:

a the range

b the mode

c the median

d the upper quartile

e the mean [8]

Question 9

Two Grade 7 classes recorded their mass as listed below:

Mass of Grade 7 boys	Frequency
36–40	3
41–45	8
46–50	12
51– 55	20
56–60	10
61–65	6
66–70	1

a Write down the class boundaries of the modal class.

b On the graph paper provided draw a cumulative frequency curve.

c Use your curve to estimate

  i the median

  ii the 70th percentile [11]

[100]

End-of-year sample exam paper: Paper 1

Question 1

a Showing all work, express $0.5\dot{7}$ as a fraction $\frac{a}{b}$ where $a, b \in Z$. (4)

b Simplify each of the following algebraic expressions:

i $(5a^2)^3$ (2)

ii $5(a^2)^3$ (2)

iii $\frac{12a^6b^6}{-4a^{-2}b^3}$ (3)

c Without using a calculator, explain fully how you would determine between which two integers $\sqrt[3]{90}$ lies. (3)

[14]

Question 2

Given: 5; 8; 11; 14; ...

a What are the next three terms? (1)

b Determine the rule for the general term of the row. (3)

c Determine the 49th term. (2)

[6]

Question 3

a Shrek decides to invest his hard-earned bonus of R1 005 000 from playing in the Rugby World Cup immediately. The recent interest rate at The People's Bank is 6,5% simple interest per year. For how many years must he invest his money in order to have at least R2,2 million to buy Fiona the engagement ring she would like to have? (3)

b Ben inherited R8 000 from his grandparent's estate. In 3 years, Ben needs R10 000 to pay for college expenses. What rate of compound interest is necessary for the inheritance to grow to the required amount? (4)

c On January 1, 2007 a Kia Sorento had a value of R320 000. Each year after that, the car's value will decrease 20% of the previous year's value. What is the value of the car on January 1, 2011? (4)

d In South Africa, the cost of a new Honda Civic is R161 300. In England the same vehicle costs £11 200 and in the USA $21 300. In which country is the car the cheapest if you compare it to the South African rand?
Note the following exchange rates: £1 @ R14,70 and $1 @ R7,30 (3)

[14]

Question 4

a Remove the brackets and simplify: $(2d + 3)(4d^2 - 8d + 9)$ (3)

b Factorise the following expressions:

i $14n^2 - 5n - 1$ (2)

ii $x^2 - 2x + 1 - 9a^2$ (3)

c Simplify as far as possible:

$\frac{4n^2 - 9}{2n + 3} \div \frac{n + 1}{2n^2 - n - 3}$ (3)

[11]

Question 5

a Solve the following equations:

i $x(x + 3) = 10$ (3)

ii $2^{3x - 1} = 4^{x + 5}$ (3)

b Solve for x:

i $\frac{x - 1}{4} \geq 5$ (2)

ii $2 \leq 3x - 5 < 9$ (3)

iii $3T = ax^6$ (3)

c Solve for x and y in the following simultaneous equations:

$4x + 3y = 7$ and $6x - 2y = -9$ (5)

[19]

Question 6

a Given the functions $f(x) = -x^2 + 4$ and $g(x) = -2x + 4$.

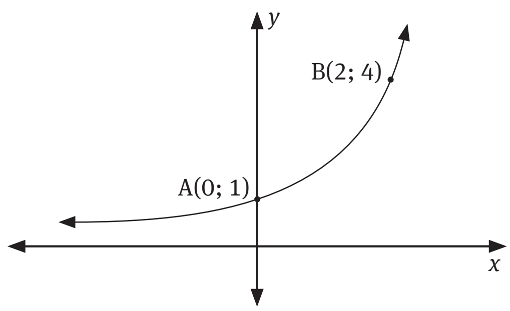
i Draw these functions on the same set of axes. (4)

ii Use the graph to determine for which values of x is $f \geq g$. (2)

iii Find the equation of the reflection of f in the x-axis. (2)

iv Find the equation of the line parallel to g and passing through $x = -2$. (2)

b The curve of the exponential function f in the accompanying diagram cuts the y-axis at the point A(0; 1), and B(2; 4) is on f.



Determine:

i The equation of the function f (2)

ii The equation of h, the reflection of f in the x-axis. (2)

iii The range of h. (1)

[15]

Question 7

a i Make a neat sketch of the graph f: $y = 3.2^x + 2$, clearly labelling intercepts with axes and any asymptotes. (3)

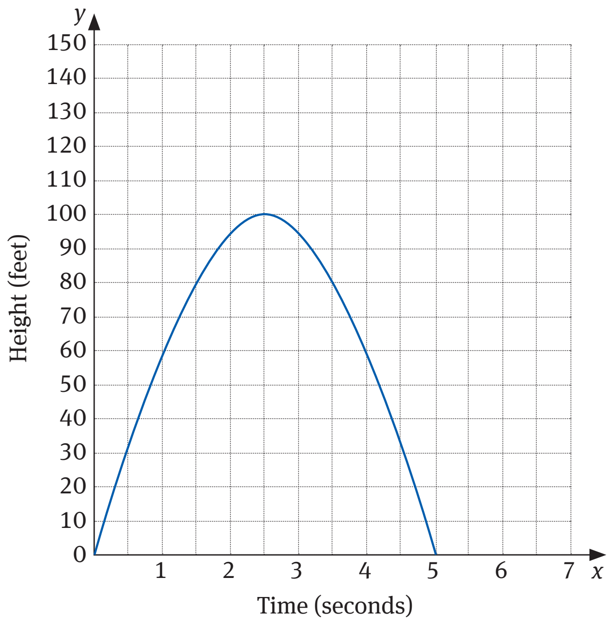
ii Write down the equation of the asymptote of f. (1)

b i Sketch the graph of $h(x) = \frac{4}{x}$; $x > 0$ (2)

ii Give the domain of h. (1)

iii Give the equation of k, the reflection of h about $y = -x$. (2)

c The following graph shows the path of the a golf ball.



i What is the range of this function? (2)

ii After how many seconds does the ball reach its maximum height ? (1)

iii What is the maximum height that the golf ball travelled in metres given that 1 foot = 30,48 cm? (2)

iv For how long was the golfer's ball airborne? (1)

d John had 15 coins in R5 and R2 pieces. He had 3 more R2 coins than R5 coins. He wrote a system of equations to represent this situation, letting x represent the number of R5 coins and y represent the number of R2 coins. Then he solved the system by graphing the equations.

i Write down the system of equations. (2)

ii Draw the graphs on the same set of axes. (2)

iii What is the solution? (2)

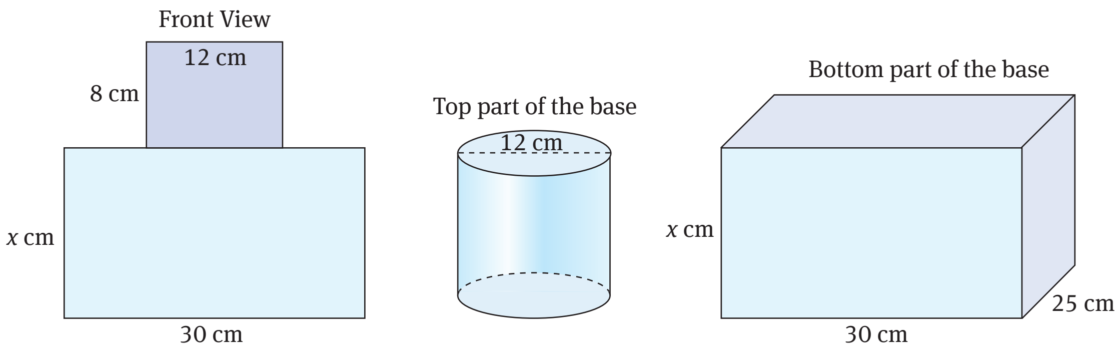
[21]

[100]

End-of-year sample exam paper: Paper 2

Question 1

a This is the sketch of the base of the Webb Ellis trophy (World Cup Rugby Trophy).



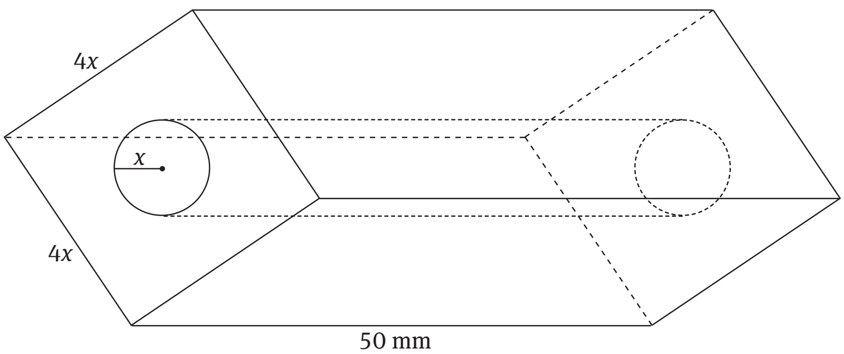
i Calculate the volume of the cylindrical piece. (2)

ii Calculate the surface area of the cylindrical piece. (3)

iii What is the effect on the volume of the entire base if the radius of the cylindrical piece is doubled? (2)

b A rectangular prism has its length, width and height doubled in length. What effect does this have on the volume of the prism? (2)

c A cylindrical hole is drilled through the centre of a rectangular metal prism, as illustrated. The cross-section of the prism is a square. The radius, x mm, of the hole is a quarter of the length of the cross-section and the height of the prism is 50 mm.



i Show that the formula for the volume, V, of the remaining metal is:
$V = 50x^2(16 - \pi)$ mm^2 (3)

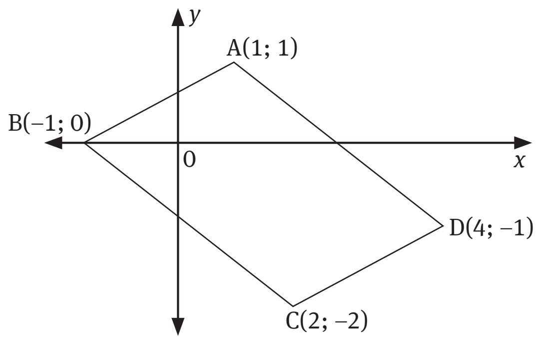
ii Calculate V if $x = 20$ mm. (2)

[14]

[5]

Question 2

Given A(1; 1), B(–1; 0), C(2; –2) and D(4; –1).



a Calculate:
- i Gradient of AD (2)
- ii Gradient of BC (2)
- iii What can you deduce from your calculations in ii and iii? (1)

b If E is the mid-point of AD, calculate the coordinates of E. (2)

c Find the equation of EF if F is on BC and EF is perpendicular on line AD. [3]

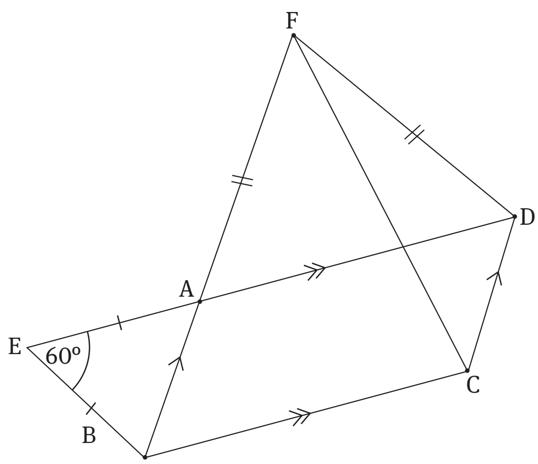
d Calculate the length of line CD. Leave your answer in surd form. (3)

e Is it possible for ABCD to be a rhombus? Justify your answer with suitable calculations. (3)

[13]

Question 3

In the diagram alongside, AF = FD, AE = EB and $A\hat{E}B = 60°$. ABCD is a parallelogram.



Prove that:

a $F\hat{D}C = EBC$ (5)

b $\Delta EBC \equiv \Delta CDF$ (8)

[13]

Question 4

a You are given that $\tan \theta = \frac{5}{12}$. Find, using a suitable diagram:

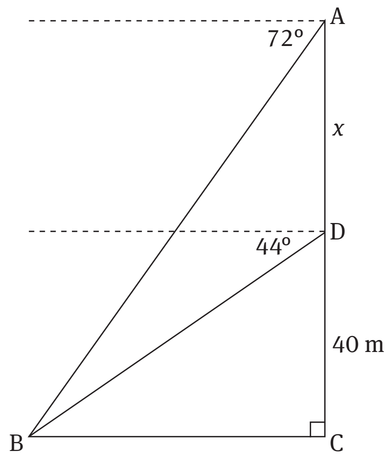
i $\cos \theta$ (3)

ii $\tan \theta . \sin \theta$ (2)

iii θ (2)

b In ΔABC, $\angle A = 43°$; $\angle B = 90°$ and AC = 11 cm. Find the area of the triangle. (5)

c From the external lift at the top of the Sun Hotel at A, the angle of depression to the entrance to an underground shopping complex across the street, at B, is 72°.
After the lift has descended x metres to D (40 m above C), the angle of depression to B, is 44°.



i Calculate the width of the street, BC. (2)

ii Calculate the height of the hotel, AC. (3)

d Using the yellow diagram sheet provided, draw a sketch of the graph of: $y = 2 \sin x$; $x \in [0°; 360°]$. Show all your critical points. (3)

[20]

Question 5

a You have entered a Blackjack competition at a casino. The dealer deals you one card, you look at it. Then he deals you another card. (Using a normal deck of cards, each card (2 to 10) represents its numerical value, with the face cards (Jack, Queen, King) value = 10, and Ace = 11.
Calculate the probability of:

i The first card being a diamond (1)

ii Getting two aces (3)

iii Getting Blackjack (the value of the two cards = 21) (5)

b When you go on holiday, the probability that it will be a warm day is $\frac{2}{3}$. If it is a warm day, the probability that you will eat ice-cream is $\frac{5}{6}$. If it is not a warm day, the probability that you will not have ice-cream is $\frac{1}{2}$. The owner of the ice-cream shop wants to know the probability that you eat ice-cream when you go on holiday to the town, regardless of the temperature. Draw a tree diagram to help the owner answer the question. (6)

[15]

Question 6

A survey was conducted at Newlands Cricket Stadium, where 22 spectators were asked how much money they spent on food and beverages during a one-day international. The results were as follows (amount shown in rand).

18	19	24	24	30	32	32	4
41	41	41	50	51	57	58	75
78	78	80	87	94	100		

a Using any method, calculate (show all your workings):
   - i The range (2)
   - ii The mode (1)
   - iii The median (2)
   - iv The mean (3)
   - v The semi-interquartile range (4)

b The heights of 80 Grade 10 Boys were recorded as follows:

Height (cm)	Number of Boys
131–140	1
141–150	4
151–160	10
161–170	20
171–180	34
181–190	7
191–200	3
201–210	1

   - i Draw a cumulative frequency curve representing the above data. (5)
   - ii Use your graph to estimate the median and the upper quartile (3)
   - iii Only 15% of the boys were tall enough to play lock for the 1st Rugby team. How tall do you have to be to qualify to play lock? (2)

[22]

[100]

Answers

Answers to questions

Chapter 1

1 a $(3x-4)^2 = 9x^2 - 24x + 16$

$9x^2$ is the square of $3x$

$-24x$ is twice the product of $3x$ and -4

16 is the square of -4

b $4x^2 - 8x - 5$

c $8a^3 + b^3$

d $(2a-5b)(4a-3b) - (a+3b)(5a-12b)$

$= (8a^2 - 26ab + 15b^2) - (5a^2 + 3ab - 36b^2)$

$= 3a^2 - 29ab + 51b^2$

e $\left(k+\frac{3}{4}\right)\left(k-\frac{1}{2}\right) = k^2 - \frac{1}{2}k + \frac{3}{4}k - \frac{3}{8} = k^2 + \frac{1}{4}k - \frac{3}{8}$

2 a $(5x+3)^2$

b $(5x+3)(x-2)$

c $x^3 - \frac{1}{4}x = x\left(x^2 - \frac{1}{4}\right) = x\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)$

d $(2t-5)^3 - (2t-5)^2 = (2t-5)^2\,(2t-5-1)$

$= (2t-5)^2\,(2t-6)$

$= 2(2t-5)^2\,(t-3)$

e $3x^3 + x^2 - 3x - 1 = x^2(3x+1) - (3x+1)$

$= (x^2-1)(3x+1)$

$= (x+1)(x-1)(3x+1)$

f $m^3 - m^2 - mn^2 + n^2 = (m^3 - m^2) - (mn^2 - n^2)$

$= m^2(m-1) - n^2(m-1)$

$= (m-1)(m^2-n^2)$

$= (m-1)(m-n)(m+n)$

g $(y+3)(2y-3) + (y-1)(3-2y)$

$= (y+3)(2y-3) - (y-1)(2y-3)$ (**Note!** $(3-2y) = -(2y-3)$)

$= (2y-3)(y+3-y+1)$

$= (2y-3)(4)$

$= 4(2y-3)$

h $x^2(2x-1) - 2x(2x-1) - 3(2x-1)$

$= (2x-1)(x^2-2x-3)$

$= (2x-1)(x-3)(x+1)$

> Reminder: How to invert a fraction
> $\frac{5}{8} \rightarrow \frac{1}{\frac{5}{8}} = 1 \div \frac{5}{8} = 1 \times \frac{8}{5} = \frac{8}{5}$
> Switch the numerator and the denominator

3 a $\frac{x^2 - x - 6}{x^2 - 4x + 3} = \frac{(x-3)(x+2)}{(x-3)(x-1)} = \frac{x+2}{x-1}$ $\left[\frac{(x-3)}{(x-3)} = 1 \text{ and the factors 'cancel out'.}\right]$

b $\frac{4x^3 - 9x^2}{4x^3 + 6x^2} = \frac{x^2(4x-9)}{2x^2(2x+3)} = \frac{4x-9}{2(2x+3)}$

c $\frac{x^2 - 7x + 12}{4 - x} = \frac{(x-4)(x-3)}{-(x-4)} = -1(x-3) = 3 - x$

d $\frac{a(a+1) + (a+1)}{a^2 - 2a + 1} \times \frac{a^2 - 1}{a^2} \div \frac{a^2 + a}{a^2 - a} = \frac{(a+1)(a+1)}{(a-1)(a-1)} \times \frac{(a+1)(a-1)}{a^2} \times \frac{a(a-1)}{a(a+1)} = \frac{(a+1)^2}{a^2}$

e $\frac{1}{x-2} + \frac{-4}{(x+2)^2} - \frac{1}{x+2} = \frac{(x+2)^2 - 4(x-2) - (x+2)(x-2)}{(x-2)(x+20)^2}$

$= \frac{x^2 + 4x + 4 - 4x + 8 - x^2 + 4}{(x-2)(x+2)^2}$

$= \frac{16}{(x-2)(x+2)^2}$

f $\frac{p-3}{p^2 - p - 12} + \frac{2}{3+p} - \frac{3}{8-2p} = \frac{p-3}{(p-4)(p+3)} + \frac{2}{p+3} + \frac{3}{2(p-4)}$

$= \frac{2(p-3) + 4(p-4) + 3(p+3)}{2(p+3)(p-4)}$

$= \frac{2p - 6 + 4p - 16 + 3p + 9}{2(p+3)(p-4)}$

$= \frac{9p - 13}{2(p+3)(p-4)}$

Chapter 2

1 a $2^2a^{2(n-2)} = 4a^{2n-4}$

b $5.1 + (2^3)^{-\frac{2}{3}} - (2^{-1})^{-2}.1$

$= 5 + 2^{-2} - 2^2$

$= 4 + \frac{1}{4} - 4$

$= \frac{5}{4}$

c $\frac{2a^{-2}3^a}{6^a} = \frac{2 \times 3^a}{a^2(2 \times 3)^a}$

$= \frac{2^1 \times 3^a}{a^2 \times 2^2 \times 3^a}$

$= \frac{2^{1-2} \times 3^{a-a}}{a^2}$

$= \frac{2^{-1} \times 3^0}{a^2}$

$= \frac{1}{2a^2}$

d $\frac{12x^{2+4} - (-3)^2(x^3)^2}{3x^3} = \frac{12x^6 - 9x^6}{3x^3} = \frac{3x^6}{3x^3} = x^{6-3} = x^3$

e $\frac{27^{x+1}.9^{x-1}}{3^{x-1}.81^{x+1}} = \frac{3^{3(x+1)}.3^{2(x-1)}}{3^{x-1}.3^{4(x+1)}}$

$= 3^{3x+3+2x-2-x+1-4x-4}$ [Be careful with negatives!]

$= 3^{-2}$

$= \frac{1}{9}$

f $\frac{2^{-3}x}{3^{-1}x^{-2}} = \frac{3x.x^2}{2^3} = \frac{3x^3}{8}$

g $\frac{4^{a-1}.8^{a+1}}{2^{a-2}.16^{a+1}} = \frac{2^{2(a-1)}.2^{3(a+1)}}{2^{a-2}.2^{4(a+1)}}$

$= 2^{2a-2+3a+3-a+2-4a-4}$

$= 2^{-1}$

$= \frac{1}{2}$

h $\frac{1}{8^{-\frac{2}{3}}} - 3a^0 + 27^{\frac{1}{3}} - 1^{\frac{2}{3}} = \frac{1}{(2^3)^{-\frac{2}{3}}} - 3 \times 1 + (3^3)^{\frac{1}{3}} - 1$

$= \frac{1}{2^{-2}} - 3 + 3 - 1$

$= 2^2 - 1$

$= 4 - 1$

$= 3$

i $\frac{x^{2a-b}}{x^{b-2a}} \div \frac{x^{4a}}{x^{2b}} = \frac{x^{2a-b}}{x^{b-2a}} \times \frac{x^{2b}}{x^{4a}}$

$= x^{2a-b+2b-b+2a-4a}$

$= x^0$

$= 1$

2 a $b = 5^{n-3-n-1}$

$= 5^{-4}$

$= \frac{1}{5^4}$

b $2.25^{x+1} = 2.\ 5^{2(x+1)}$ $[25 = 5^2]$

$= 2.5^{2x+2}$

$= 2.5^{2x}.5^2$

$= 2.5^2.5^{2x}$

$= 2.25.5^{2x}$

$= 50.(5^x)^2$

$= 50f^2$

c i $3^{m+2} = 3^m.3^2 = 5 \times 9 = 45$

ii $9^{-2m} = 3^{2(-2m)} = 3^{m(-4)} = 5^{(-4)} = \frac{1}{5^4} = \frac{1}{625}$

d $(3^{2x} - 3)(3^{2x} + 3) = (3^{2x})^2 - 3^2 = 3^{4x} - 9$

If we let $3^{2x} = a$ and $3 = b$, then we have $(a - b)(a + b)$, which, when multiplied out, gives us the difference between two squares $(a^2 - b^2)$.

3 a $2^x + 1 = 2$

$2^x = 2 - 1$

$2^x = 1 = 2^0$

Therefore $x = 0$

b $9^{x-2} = 27^{1-2x}$

$3^{2(x-2)} = 3^{3(1-2x)}$

The bases are the same, therefore:

$2(x - 2) = 3(1 - 2x)$

$2x - 4 = 3 - 4x$

$2x + 4x = 3 + 4$

$6x = 7$

$x = \frac{7}{6}$

c $5^{2x+1} = 0{,}04$ [**Note:** $0{,}04 = \frac{4}{100} = \frac{1}{25} = 5^{-2}$]

$5^{2x+1} = 5^{-2}$

Therefore:

$2x + 1 = -2$

$2x = -3$

$x = -\frac{3}{2}$

d $5^{x+1} + 5^{x} - 150 = 0$

$5^{x}(5 + 1) = 150$

$5^{x} = 25[25 = 5^{2}]$

Therefore, $x = 2$

4 $(3 \times 10^{8})^{3} = 3^{3} \times (10^{8})^{3}$

$= 27 \times 10^{8 \times 3}$

$= 27 \times 10^{24}$

$= 27 \times 10 \times 10^{24}$

$= 27 \times 10^{24 + 1}$

$= 27 \times 10^{25}$

Therefore, the cubic region of space has a volume of $2{,}7 \times 10^{25}$ cubic miles.

Chapter 3

1 a 33; 40 (pattern is +7)

b $\frac{5}{9}; \frac{6}{11}$ (pattern in numerator + 1; denominator consecutive odd numbers)

2 a Common difference = –3;

$a = 25$

General term: $T_n = -3n + 28$ and $T_{16} = -3(16) + 28 = -20$

b $T_n = (2)^{n-1}$

$T_{16} = (2)^{16-1} = 32\,768$

3 a 13 matches

b 16 matches

c 4; 7; 10; 13; ... difference is +3

Therefore: $T_n = 3n + 1$

d $T_{100} = 3(100) + 1 = 301$

4 A table of values will help to see a pattern more easily:

# Cups	1	2	3	4	5	6		n
Height in cm	7	9	11	13	15	17	22	

a 13 cm

b 7; 9; 11; 13; 15; 17; ...

c $T_n = 2n + 5$

d $22 > 2(n) + 5$ (The maximum height of shelf is 22 cm)

$n < 17 \div 2$

$n = 8$ cups

e $81 = 2(n) + 5$

$n = 38$

Theoretically you can stack 38 cups to reach a height of 81 cm.

Chapter 4

1 a $\frac{5a}{3} - 2 = \frac{a}{4} + 15$

$\therefore 4(5a - 6) = 3(a + 60)$

$\therefore 20a - 3a = 24 + 180$

$\therefore 17a = 204$

$\therefore a = 12$

b $\frac{m}{4} + 15 < \frac{5m}{3} - 2$

$\therefore 3m + 180 < 20m - 24$

$\therefore -17m < -204$

$\therefore m > 12$

Note:
the inequality sign will swap over, because we are dividing by a negative number.

c $(2b + 1)(b + 8) = 27$ — Step 1: remove brackets

$2b^2 + 17b + 8 = 27$

$2b^2 + 17b - 19 = 0$ — Factorise when a quadratic equation is = 0.

$(2b + 19)(b - 1) = 0$

$b = 1$ or $b = -\frac{19}{2}$

d $\frac{2d - 2 - 1}{d - 1} = \frac{3}{d + 1}$

$(2d - 3)(d + 1) = 3(d - 1)$

$\therefore 2d^2 - d - 3 = 3d - 3$

$\therefore 2d^2 - 4d = 0$

$\therefore d(d - 2) = 0$

$\therefore d = 0$ or $d = 2$

e $b^2 - 7b + 12 = 0$

$(b - 3)(b - 4) = 0$

$b = 3$ *or* $b = 4$

f $\frac{5y-2}{3} + \frac{3y-1}{2} - \frac{y+1}{2} < -\frac{1}{3}$

$\frac{10y - 4 + 9y - 3 - 3y - 3}{6} < -\frac{1}{3}$

$\frac{16y - 10}{6} < -\frac{1}{3}$

$\frac{8y-5}{3} < -\frac{1}{3}$

$\therefore 24y - 15 < -3$

$\therefore 24y < 12$

$\therefore y < \frac{1}{2}$

g Restrictions: $h \neq \pm\frac{4}{3}; -3$

$\frac{3}{9h^2 - 16} - \frac{5}{3h^2 + 5h - 12} = \frac{2}{h(3h+4) + 3(4+3h)}$

$\frac{3h + 9 - 15h - 20 - 6h + 8}{(3h+4)(3h-4)(h+3)} = 0$

$\therefore -18h - 3 = 0$

$\therefore h = -\frac{1}{6}$

2 $2a = 24 + 7b$ .... (1)

$3a + 5b = 5$ .... (2)

(1): $a = \frac{24 + 7b}{2}$

(2): $\frac{3(24 + 7b)}{2} + 5b = 5$

$72 + 21b + 10b = 10$

$\therefore 31b = -62$

$\therefore b = -2$

and $a = \frac{24 + 7(-2)}{2} = 5$

3 $3V = 4\pi R^3 - 4\pi r^3$

$4\pi R^3 = 3V + 4\pi r^3$

$R = \sqrt[3]{\frac{3V + 4\pi r^3}{4\pi}}$

4 a $x^2 + 3(2x - 5) = x^2 - 2(7 + 3x)$

$12x = 1$

$\therefore x = \frac{1}{12}$

b $y - 3x = 2x + 2z$

$-6x = 2z - y$

$\therefore x = \frac{2z - y}{-6}$

or $x = \frac{y - 2z}{6}$

c $x + 5 + 6 = 6x - 10 + 2x$

$\therefore -7x = -21$

$\therefore x = 3$

d Restrictions: $x = \pm 3$

$\frac{3}{(x+3)(x-3)} + \frac{2}{x+3} = \frac{1}{x-3}$

$\therefore 3 + 2x - 6 = x + 3$

$\therefore x = 6$

e $24x - 8(x - 2) \geq 36 + 21x$

$\therefore 24x - 8x + 16 \geq 36 + 21x$

$\therefore -5x \geq 20$

$\therefore x \leq -4$

5 $7a + 14 + 3b - 15 = 34$

$7a + 3b = 35$ .... (1)

and $3a + 6 - 2b + 10 = 8$

$3a - 2b = -8$ .... (2)

(1) $\times 2$: $14a + 6b = 70$ .... (3)

(2) $\times 3$: $9a - 6b = -24$ .... (4)

(3) + (4): $23a = 46$

$a = 2$

$b = 7$

Chapter 5

1 a In ΔADC:

$\tan A = \frac{CD}{AD}$

$AD = \frac{CD}{\tan A} = \frac{15}{\tan 55°}$

AD = 10,50 cm

AB = 2 × 10,50 cm [AD = BD]

= 21 cm

b In ΔADC:

$\sin A = \frac{CD}{AD}$

$AC = \frac{CD}{\sin A}$

$AC = \frac{15}{\sin 55°}$

AC = 18,31 cm

c In ΔACD:

$\angle ACD = 90° - 55°$ [$\angle$s of Δ = 180°]

$= 35°$

In ΔDBC:

$\angle DCB = 35°$ [$\Delta ADC \equiv \Delta BDC$]

$\cos \angle DCB = \frac{CD}{BC}$

$BC = \frac{CD}{\cos D\hat{C}B}$

$BC = \frac{15}{\cos 35°}$

BC = 18,31 cm [ΔABC is isosceles]

d $\angle DBC = \angle DAC = 55°$

2 a Speed = distance ÷ time

Therefore, distance = speed × time

speed = 6 000 km/h

2 min = $\frac{2}{60}$ hour

Therefore, distance (AD) = 6 000 × $\frac{2}{60}$ = 200 km

b BE (height after 3 min) = BC (BC = AD = 200 km) + CE

CE is opposite $\angle$EDC, so we need $\angle$EDC

$\angle EDC = 90° - 55° = 35°$

$\sin 35° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{CE}{DE}$

Therefore, CE = DE × sin 35°

DE = 6 000 km/h × $\frac{1}{60}$

DE = 100 km

Therefore, CE = 100 km × sin 35° = 57,36 km

Therefore, BE = 200 km + 57,36 km = 257,36 km

c AB = DC

$\cos 35° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{DC}{DE}$

DC = DE cos 35° and DE = 100 km (from b)

Therefore, AB = 100 km × 0,85264

AB = 85,26 km (to two decimal places)

3 You have two right-angled triangles that you can use to solve the problem: ΔABC and ΔABD. To determine CD, find the distance BC and subtract it from distance BD: CD = BD – BC. $\angle$AEC is alternate and equal to $\angle C_1$ (AE and BD are parallel lines) and equal to 25°. The side opposite $\angle C_1$ is AB, and BC is adjacent to $\angle C_1$. So the tan-ratio is suitable.

$\angle C_1 = 25°$ and AB = 60

$\frac{AB}{BC} = \tan 25°$

$BC = \frac{60}{\tan 25°}$

BC = 128,6704

To determine BD, we use ΔABD. ∠AED is alternate and = ∠ADB (AE and BD are parallel lines) = 20°. The side opposite ∠ADB is AB and BD is adjacent to ∠ABD. So again the tan ratio is suitable.

∠ADB = 20° and AB = 60

$\frac{AB}{BD} = \tan 20°$

$BD = \frac{60}{\tan 20°}$

BD = 164,8486

The distance between the ships C and D = BD – BC = 36,2 metres.

4 a ΔMPN = 180° – 52° – 38° = 90° [Angles in a triangle add up to 180°]

b $\sin 38° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{MP}{MN}$

MP = 160 × sin 38°

MP = 99 m

Or

$\cos 52° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{MP}{MN}$

MP = 160 × cos 52°

MP = 99 m

c $\sin 52° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{PT}{MP}$

PT = MP × sin 52°

PT = 98,5 × 0,788

≈ 78 m

5 a $x = 34{,}75°$

b $\cos x = \frac{0{,}96}{3}$

$x = 71{,}34°$

c $2\tan x = 4{,}2 \times 3$

$\tan x = 4{,}2 \times 3 \div 2$

$x = \tan^{-1}(4{,}2 \times 3 \div 2) = 80{,}98°$

d $\sin(x + 25°) = 0{,}813$

$x + 25° = \sin^{-1}(0{,}813) = 54{,}39°$

$x = 54{,}39° - 25° = 29{,}39°$

e $\tan x = 0{,}6691$

$x = 33{,}79°$

6 a $\sin x = \frac{\sqrt{3}}{2}$
$x = 60°$

b $3x = 60°$
$x = 20°$

c $4\sin(2x - 10°) = 2$
$\sin(2x - 10°) = \frac{2}{4}$
$(2x - 10°) = \sin^{-1}\frac{1}{2}$
$2x = 30° + 10° = 40°$
$x = 20°$

d $x = \frac{\cos 30°}{\tan 60°}$
$= \frac{\sqrt{3}}{2} \div \frac{\sqrt{3}}{1}$
$= \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}}$
$= \frac{1}{2}$

Chapter 6

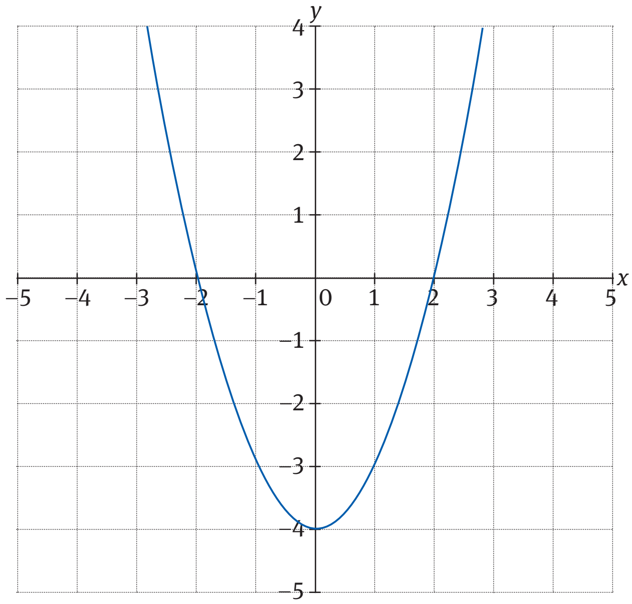
1 a The graph is a parabola.

b Minimum value: $[a = 1 > 0]$

c $y = -4$ (Note that just -4 would not be sufficient)

d For $y = x^2 - 4$, let $y = 0$:
$0 = x^2 - 4$
$4 = x^2$
$x^2 = 4$
$x = \pm 2$

e x-intercepts: $(2; 0)$ and $(-2; 0)$
minimum value at $y = -4$
So you have three points $(-2; 0)$, $(2; 0)$ and the turning point $(0; -4)$. The graph is a parabola and a is positive, therefore the “arms” point upwards.



f $y = x^2 - 4 + 4$

$y = x^2$

g The x-intercepts are equal; they are $x = 0$.

h Domain: all real values of x ($x \in$ R),

Range: all y-values greater or equal to −4

$y \geq -4$, $y \in$ R or $[-4; \infty)$

i The y-axis is the axis of symmetry ($x = 0$).

2 a The general equation is: $y = ax^2 + q$ and the turning point is (0; 3).

Replace x and y with the values of the coordinates of the turning point:

$y = ax^2 + 3$

Substitute(−3; 0) into the equation:

$0 = a(-3)^2 + 3$

$0 = 9a + 3$

$a = \frac{-3}{9} = \frac{-1}{3}$

$y = x^2 + 3$

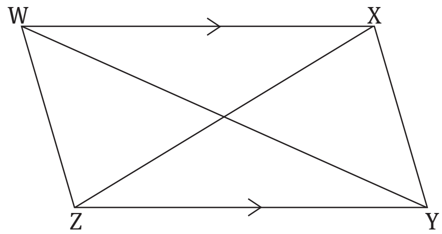
b Domain: $x \in$ R; Range: $y \leq 3$, $y \in$ R (or domain: $(-\infty; \infty)$ range: $[-\infty; 3)$)

c $x = 0$ (**note:** "y-axis" is not sufficient, because an equation is asked for)

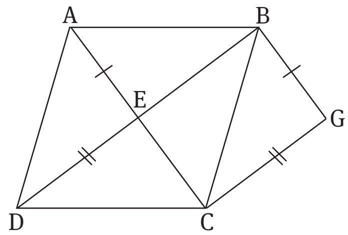
Chapter 7

1. ΔWXZ = ΔWYZ(given)

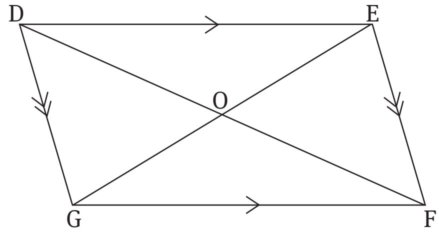
∴ XY || WZ	(= Δs on same base; ||)
And WX || ZY	(given)
∴ WXYZ is a ||m	(opp sides ||)



2 BG || AE (given)
∴ BG || EC
CG || DE (given)
∴ CG || EB
∴ BGCE a ||m (2 pr opp sides ||)
And ∆BEC = 90° (diags of rhomb ⊥)
∴ BGCE a rect (||m with int ∠ = 90°)



3 DEFG is a parm (both pairs opp sides ||)
∆GDO = ∆EFO (DG || EF)
∆DGO = ∆FEO (DG || EF)
DG = EF (opposite sides of parm)
∴ ∆DGO ≡ ∆FEO (AAS)
∴ DO = OF and GO = OE



4 a In ∆ADE and ∆ABC:
AD : AB = 1 : 2 (AD = DB)
$\hat{A} = \hat{A}$ (common)
AE : AC = 1 : 2 (AE = EC)
∴ ∆ADE ||| ∆ABC (SAS)
b $A\hat{D}E = \hat{B}$ (∆ADE ||| ∆ABC)
∴ DE || BC (corresponding ∠s equal)
c DE : BC = AD : AB = 1 : 2 (∆ADE ||| ∆ABC)
∴ DE = $\frac{1}{2}$BC

Chapter 8

1 a You need the gradient formula:
$m_{PQ} = \frac{y_P - y_Q}{x_P - x_Q} = \frac{-1 - 3}{-4 - 6} = \frac{-4}{-10} = \frac{2}{5}$
b Remember, if lines are parallel, their gradients are equal ($m_{PQ} = m_{SR}$), therefore:
$m_{SR} = \frac{y_S - y_R}{x_S - x_R} = \frac{-3 - b}{-4 - 6} = \frac{2}{5}$
$-15 - 5b = -20$
$b = 1$
c You need the distance formula:

$PQ^2 = (x_p - x_q)^2 + (y_p - y_q)^2$
$PQ^2 = (-4 - 6)^2 + (-1 - 3)^2$
$PQ^2 = 100 + 16$
$PQ^2 = 116$

$SR^2 = (x_s - x_r)^2 + (y_s - y_r)^2$
$SR^2 = (-4 - 6)^2 + (-3 - 1)^2$
$SR^2 = 100 + 16$
$SR^2 = 116$

Therefore, PQ = SR

d Check the properties of a parallelogram!
PQ = PS (just proven) and PS || QR (gradients are the same)
∴ PQRS parallelogram (1 pair opposite sides equal and parallel)

e i M_{PR}: $\left(\frac{x_p + x_r}{2}; \frac{y_p + y_r}{2}\right) = \left(\frac{-4+6}{2}; \frac{-1+1}{2}\right) = (1; 0)$

ii M_{SQ}: $\left(\frac{x_s + x_q}{2}; \frac{y_s + y_q}{2}\right) = \left(\frac{-4+6}{2}; \frac{-3+3}{2}\right) = (1; 0)$

f They bisect each other at the point (1; 0).

g $PS^2 = (x_p - x_s)^2 + (y_p - y_s)^2$
$PS^2 = (-4 - (-4))^2 + (-1 - (-3))^2$
$PS^2 = 0 + 4$
$PS^2 = 4$
∴ PS = 2 units and PQ = $\sqrt{116}$ = 10,77 units
∴ PQ ≠ PS
∴ PQRS is not a rhombus, because the adjacent sides are not equal.

2 a $AB^2 = (-4 - 2)^2 + (-3 - 5)^2$
$= (-6)^2 + (-8)^2$
$= 36 + 64 = 100$
∴ AB = $\sqrt{100}$
∴ AB = 10 units

b K = (−1; 1)

c $m = \frac{y_2 - y_1}{x_2 - x_1}$
$= -\frac{8}{6} = -\frac{4}{3}$

3 $m_{AB} = m_{DC}$
$m_{DC} = \frac{5-0}{7-6} = 5$
∴ $x = 1$ and $y = -2$

4 a $AB^2 = (5 - 3)^2 + (11 - 7)^2$
$= 4 + 16 = 20$
AB = $\sqrt{20}$ units

b $BC^2 = (6 - 5)^2 + (3 - 11)^2$
$= 1 + 64 = 65$
BC = $\sqrt{65}$ = 8,06 units

c Midpoint = $\left(\frac{3+6}{2}; \frac{7+3}{2}\right)$
$= (4\frac{1}{2}; 5)$

d $m_{AB} = \frac{11-7}{5-3} = 2$ and $m_{BC} = \frac{11-3}{5-6} = -8$

5 $m_{AB} = \frac{6-3}{2+2} = \frac{3}{4}$

$m_{BC} = \frac{k-6}{8-2} = \frac{k-6}{6}$

AB ⊥ BC $\therefore m_{AB} \times m_{BC} = -1$

$\therefore \left(\frac{3}{4}\right)\left(\frac{k-6}{6}\right) = -1$

$\therefore \frac{k-6}{8} = -1$

$\therefore k = -2$

6 a $HK^2 = (1+5)^2 + (1+3)^2 = 36 + 16 = 52$

$\therefore HK = \sqrt{52}$ units

b KD = HG

$\therefore KD^2 = HG^2 = (3+5)^2 + (7-1)^2 = 100$

$\therefore$ KD = 10 units

c $m_{GD} = m_{HK}$

$\therefore \frac{7-y}{3-x} = \frac{4}{-6}$

$\therefore$ D = (9; 3)

d Midpoint of HD = midpoint of KG:

$\left(\frac{3+1}{2}; \frac{7-3}{2}\right) = (2; 2)$

7 a $Q = \left(\frac{-4+3}{2}; \frac{0-7}{2}\right) = \left(-\frac{1}{2}; -3\frac{1}{2}\right)$

b $m_{MN} = \frac{0+7}{-4-3} = -1$ and $m_{PQ} = \frac{4+3\frac{1}{2}}{7+\frac{1}{2}} = 1$

c Product of gradients = (1)(−1) = −1

d MN || PQ

e $PM^2 = (7+4)^2 + (4-0)^2 = 121 + 16 = 137$

$\therefore PM = \sqrt{137}$ units

$PN^2 = (7-3)^2 + (4+7)^2 = 16 + 121 = 137$

$\therefore PN = \sqrt{137}$ units

f The triangle is a right-angled isosceles triangle, because PM = PN and there is an angle of 90° between MN and PQ.

Chapter 9

1 $A = P(1 + I)^n$
$= 70\ 000(1 + 0{,}09)^6$
$= R117397{,}01$

2 $A = P(1 + in)$
$= 30\ 000(1 + 0{,}06 \times 8)$
$= R44\ 400$

3 a Cash paid = 20% of R105 000
$= \frac{20}{100} \times \frac{105\ 000}{1}$
= R21 000 Loan amount = R105 000 – R21 000 = R84 000

b $A = P(1 + in)$
$= 84\ 000(1 + 0{,}145 \times 5)$
= R144 900 Full Amount = R144 900 + R21 000 = R165 900

c Monthly payments $= \frac{144\ 000}{84}$
= R1 725

d Total amount paid = R780 × 84 + R21 000 + 84 × R1 725 = R231 420
Or
Total amount paid = R165 900 + 84 × R780 = R231 420

4 Price of pram in SA currency = R26,09 × 80 = R2 152

5 Amount in British pounds = R530 000 ÷ 11,72 = £45 222

6 $A = P(1 + in)$
$= 420\ 000(1 + 0{,}125 \times 8)$
$= R840\ 000$

7 $A = P(1 + i)^n$
$180\ 000 = P(1 + 0{,}0725)^{10}$
$P = \frac{180\ 000}{(1.0725)10}$
$= R89\ 392$

8 $A = P(1 + i)^n$
$= 50\ 000(1 + 0{,}04)^5$
$= R60\ 833$
Percentage growth $= \frac{60\ 833 - 50\ 000}{50\ 000} \times 100$
= 21.67%

9 Amount in \$ = 40 000 × \$76 = \$3 040 000
Amount in R = 3 040 000 × R7,23 = R21 979 200

10 Option 1: Simple growth: $A = P(1 + in)$
$= 30\ 000(1 + 0{,}125 \times 8)$
$= R60\ 000$
Option 2: Compound growth: $A = P(1 + i)^n$
$= 30\ 000(1 + 0{,}125)^8$
$= R76\ 973{,}54$
He will choose option 2 because he will receive more interest on his investment.

Chapter 10

1 a 197

b KwaZulu-Natal

c The province is mainly rural. There are not too many activities in the province over the Easter weekend. Drivers and pedestrians obey the rules of the road. (Or any other meaningful reasons.)

d *percentage* $= \frac{37}{197}$

$= 18.78\%$

2 a The range = 188 cm – 134 cm = 54 cm

b The heights of the learners vary considerably (by more than half a metre).

3 a Range value for the marks of female learners = 70% – 48% = 22%

b Range value for the marks of male learners = 74% – 22% = 52%

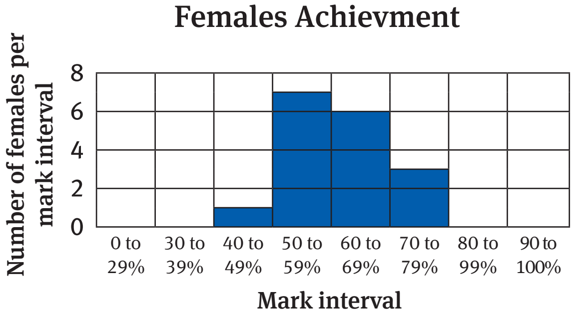
c The difference between the range value for female learners tells us that they are of similar ability, whereas the wide spread of marks for the male learners tells us that *t* here is a wide spread of ability amongst the males.

d i

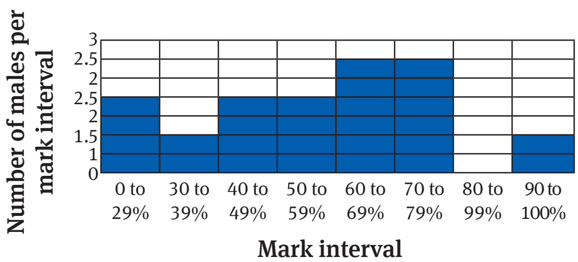
	Females	Males
Mark category	**No. of learners who achieved marks in this category**	**No. of learners who achieved marks in this category**
0 – 29%	0	2
30 – 39%	0	1
40 – 49%	1	2
50 – 59%	7	2
60 – 69%	6	3
70 – 79%	3	3
80 – 89%	0	0
90 – 100%	0	1

ii

Females Achievment



Males Achievment



iii From the two graphs we notice the following:
The female histogram tells us that learners' abilities are similar. The histogram is fairly compact.
The male histogram tells us that learners' achievements are of a very mixed ability. The histogram is very spread out.

iv It looks as though the female learners performed better because the results are grouped together higher up the scale. However, even though the male learners have the highest mark, there is a spread towards the lower end of the scale, which would lower their average.

4 a Learner A: 266 ÷ 6 = 44,33%
Learner B: 214 ÷ 6 = 35,67%
Learner C: 329 ÷ 6 = 54,83%

b Learner A: 69 – 22 = 47%
Learner B: 55 – 11 = 44%
Learner C: 64 – 45 = 19%

c Learner C is the only one of the three with a mean over 50% and is also the most consistent (range = 11%). Therefore, Learner C should receive the bursary.

Chapter 11

1 $\text{volume}_{cube} = 12^3$ and $\text{volume}_{prism} = 18 \times 8 \times \text{height}$

$\therefore$ height of water $= \frac{12^3}{18 \times 8} = 12$ cm

2 Surface area = 2 × Area of base + (perimeter of base × height)

= Surface area of cube – 2 area of circles on sides + circumference of circle × height of cylinder

$= (2 \times 10^2 + 40 \times 10) - 2 \times \pi \times \left(\frac{3}{2}\right)^2 + 2 \times \pi \times \frac{3}{2} \times 10$

$= 680{,}11 \text{ cm}^2$

3 Note that measurements are given in mm, but the answer has to be in cm^3.

Convert: 15 mm = 1,5 cm and 20 mm = 2 cm

Volume of material = Outer volume – Inner volume

$V = \frac{4}{3}\pi(2^3 - 1{,}5^3) \text{ cm}^3$

$= 19{,}4 \text{ cm}^3$

4 Note that there are two different types of units: mm and cm

So, 70 mm = 7 cm

$V = \frac{1}{3}\pi r^2 h$

$r = \sqrt{\frac{3V}{\pi h}}$

$r = \sqrt{\frac{3 \times 4\,224}{7\pi}}$

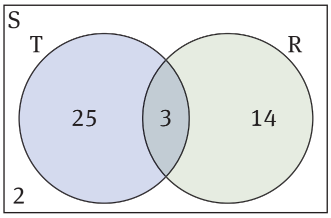
slant height $= \sqrt{h^2 + r^2}$

$s = \sqrt{7^2 + 24^2}$

$s = 25$ cm

Total surface area $= \pi \times 24(24 + 25) = 3\,694{,}5 \text{ cm}^2$

Chapter 12

1 a 

b i $n(T \cap R) = 3$

ii $P(T) = 28$

iii $P(T \cup R) = 42$

iv $n(R') = 25 + 2 = 27$

v $n(S) = 44$

vi $n(T \text{ or } R) = 28 + 17 - 3 = 42$

2 a There are 12 possible outcomes:
(H; 1); (H; 2); (H; 3); (T; 1); (T; 2); (T; 3)

b P(heads and 2) = $\frac{2}{12} = \frac{1}{6} = 0{,}17$

c P(tails and an uneven number) = $\frac{4}{12} = \frac{2}{6} = \frac{1}{3} = 0{,}3\dot{3}$

Mid-year exam paper

Question 1

a i $x > 2$ ii $x = 2$

b i $6^2 - 5 = 6 + 5^2$ and $7^2 - 6 = 7 + 6^2$

ii If m and n are consecutive integers and $n > m$, then $n^2 - m = n + m^2$.

iii $x^2 - (x - 1) = x + (x - 1)^2, x \in Z$

iv LHS $= x^2 - x + 1$

RHS $= x + x^2 - 2x + 1 = x^2 =$ LHS

conjecture proved

v $(n + 2)^2 - (n + 1) = (n + 2) + (n + 1)^2$ for $n \in$ natural numbers

Question 2

a i $\frac{2^n.2^{-2}.3^n.3^3}{2^n.3^n} = \frac{27}{4}$

ii $\frac{16x + 48 - 9x - 48}{12} = \frac{7x}{12}$

b $6x^3 + 13x^2 + 4x - 3$

c i $(x - 6)(x + 1)$

ii $4a(3n - 2) + m(3n - 2) = (3n - 2)(4a + m)$

iii $(x + 2)(x^2 - 4) = (x + 2)(x + 2)(x - 2) = (x + 2)^2(x - 2)$

Question 3

a $-2a \leq 74$

$\therefore a \geq -37$

b $2^{7 - 2a} = 8 = (2^3)$

$\therefore 7 - 2a = 3$

$\therefore a = 2$

Question 4

$6x + 4y = 10$

$6x - 9y = 36$

$\therefore 13y = -26$

$\therefore y = -2$

and $x = 3$

Question 5

a $x^2 - 7x - 8 = 0$

$\therefore (x - 8)(x + 1) = 0$

$\therefore x = -1$ or 8

b $\frac{x^2 - 4x + 3}{3 - x} = \frac{x + 1}{2}$

$1 - x = \frac{x + 1}{2}$

$x + 1 = 2 - 2x$

$x = \frac{1}{3}$

c $x = -2$ or 0

Question 6

a $V = 9 \times 7 \times 4 = 252 \text{ cm}^3$

b i $r = 0$, $\therefore$ depth = 6 m

If $r = 1$ m, $h = 5\frac{1}{4}$

ii When r decreases by 1 m, h increases by $\frac{3}{4}$ m.

iii $h = 0 \therefore 6 - \frac{3}{4}r = 0$

$\therefore$ radius = 8 m

iv $C = 2\pi r = 2\pi(8) = 16\pi = 50{,}27$ m

Question 7

a $AB = \sqrt{6^2 + 3^2} = \sqrt{45}$

b Midpoint = $\left(-\frac{1}{2}; 0\right)$

c $m_{AB} = -2$

$y = \frac{1}{2}x + c$

d $\therefore 0 = \frac{1}{2}\left(-\frac{1}{2}\right) + c$

$\therefore y = \frac{1}{2}x + \frac{1}{4}$

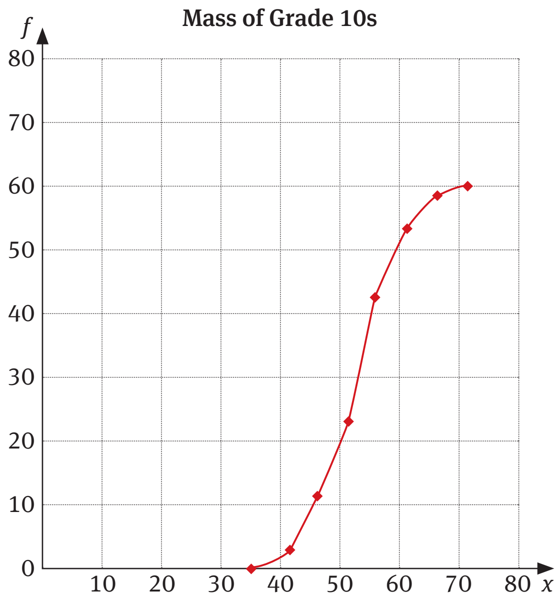
Question 8

a $31 - 2 = 29$ b 11 c 13 d 19 e $\frac{140}{10} = 14$

Question 9

a 50,5 and 55,5

b



i Between 51 and 52,5

ii Between 54,5 and 55,5

End-of-year sample exam paper: Paper 1

Question 1

a $x = 0,57777777$ .... (1)

$10x = 0,5777777$ .... (2)

(2) – (1): $9x = 5,2$

$x = \frac{5,2}{9} = \frac{52}{90} = \frac{26}{45}$

b i $125a^6$

ii $5a^6$

iii $-3a^8b^3$

c $\sqrt[3]{64} = 4$

$\sqrt[3]{125} = 5$

Therefore, between 4 and 5

Question 2

a 17; 20; 23

b $T_n = a + (n - 1)d$

$= 5 + 3(n - 1)$

$= 5 + 3n - 3$

$= 3n + 2$

c $T_{49} = 3(49) + 2 = 140$

Question 3

a $2\ 200\ 000 = 1\ 005\ 000(1 + 0,065 \times n)$

$n = \left[\frac{2\ 200\ 000}{10\ 050\ 000} - 1\right] \div 0,065 = 18,3$

For 19 years

b $10\ 000 = 8\ 000(1 + i)^3$

$(1 + i)3 = \frac{10}{8}$

$1 + i = \sqrt[3]{\frac{5}{4}}$

$i = \sqrt[3]{\frac{5}{4}} - 1$

$i = 0,0772...$

$i = 7,7\%$

c $A = 320\ 000(1 - 0,2)^4$

$A = R131\ 072$

d $11\ 200 \times 14,70 = R164\ 640$

$21\ 300 \times 7,30 = R155\ 490$

The car is cheapest in the USA.

Question 4

a $8d^3 - 4d^2 - 6d + 27$

b i $(7n + 1)(2n - 1)$

ii $(x - 1 - 3a)(x - 1 + 3a)$

c $\frac{(2n-3)(2n+3)}{2n+3} \times \frac{(2n-3)(n+1)}{n+1} = (2n-3)^2$

Question 5

a i $x^2 + 3x - 10 = 0$

$(x + 5)(x - 2) = 0$

$x = -5$ or 2

ii $2^{3x-1} = 2^{x+10}$

$\therefore 3x - 1 = 2x + 10$

Therefore $x = 11$

b i $x - 1 \geq 20$

$x \geq 21$

ii $7 \leq 3x < 14$

$\frac{7}{3} \leq x < \frac{14}{3}$

iii $x^6 = \frac{3T}{a}$

$x = \pm \sqrt[6]{\frac{3T}{a}}$

c $8x + 6y = 14$ .... (1)

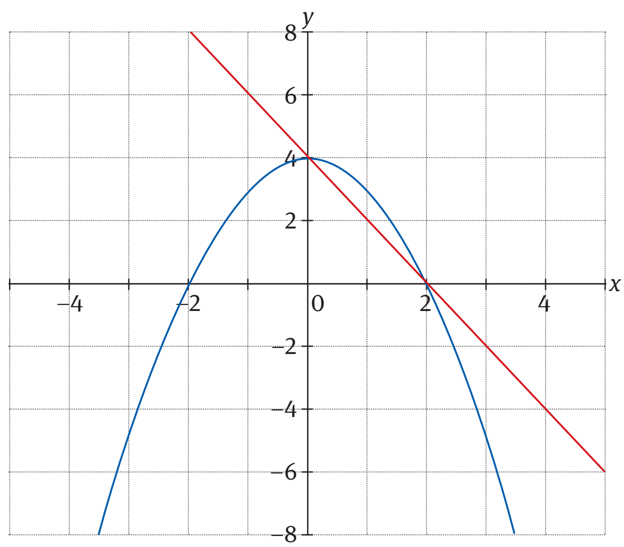
$18x - 6y = -27$ .... (2)

(2) + (1): $26x = -13$

$x = -\frac{1}{2}$ and $y = 3$

Question 6

a i



ii $0 \leq x \leq 2$

iii $y = x^2 - 4$

iv $y = -2x - 4$

b i $y = a.b^x$

$4 = 1.b^2$

$\therefore b = 2$

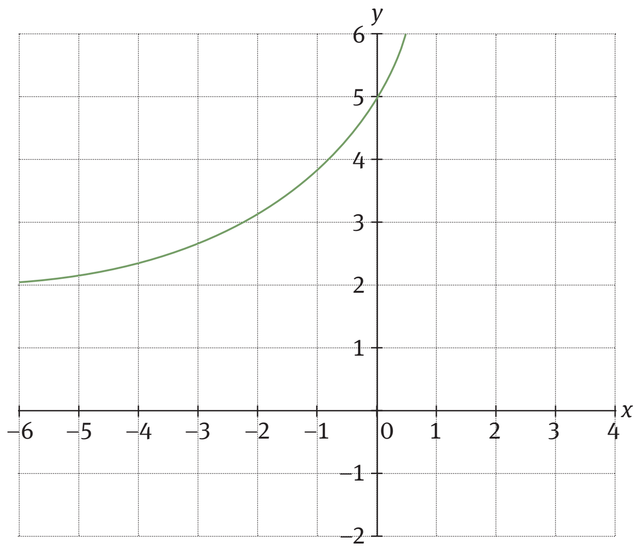
$f(x) = 2^x$

ii $h(x) = -2^x$

iii $y < 0$

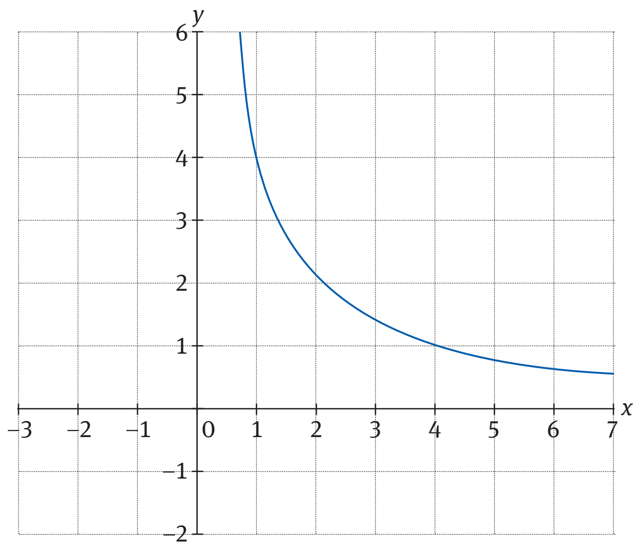
Question 7

a i



ii $y = 2$

b i



ii $x > 0$

iii $y = \frac{4}{x}; x < 0$

c i $0 \leq y \leq 100$

ii $2\frac{1}{2}$

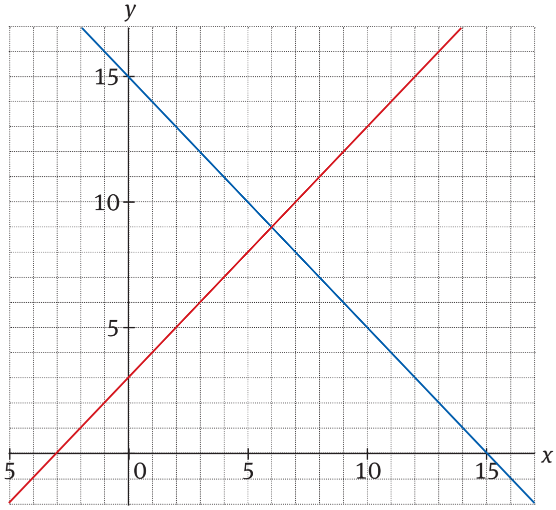
iii $\frac{100 \times 30{,}48}{100} = 30{,}48$ m

iv 5 seconds

d i $x + y = 15$

$x + 3 = y$

ii



Equation 1: $x + y = 15$

Equation 2: $x + 3 = y$

iii $x = 6$; $y = 9$

End-of-year sample exam paper: Paper 2

Question 1

a i Volume = $\pi r^2 h$
$= \pi(6)^2 \times 8$
$= 904{,}8 \text{ cm}^3$

ii $SA = 2\pi rh + 2\pi r^2$
$= 2\pi(6)(8) + 2\pi(6)^2$
$= 527{,}8 \text{ cm}^2$

iii Entire volume is 3 times bigger.

b Volume is 8 times bigger.

c i $V = (4x \times 4x \times 50) - \pi x^2 \times 50$
$= (800x^2 - 50\pi x^2) \text{ mm}^3$
$= 50x^2(16 - \pi) \text{ mm}^3$

ii $V = 800(20)^2 - 50\pi(20)^2$
$= 257\ 168{,}15 \text{ mm}^3$

Question 2

a i $m_{AD} = \frac{-1-1}{4-1} = -\frac{2}{3}$

ii $m_{BC} = \frac{-2-0}{2+1} = -\frac{2}{3}$

iii AD || BC

b Midpoint E $= \left(\frac{1+4}{2}; \frac{1-1}{2}\right) = (2{,}5; 0)$

c $m_{EF} = \frac{3}{2}$

$\therefore y = \frac{3}{2}x + c$

$\therefore 0 = \frac{3}{2}(2{,}5) + c$

$\therefore c = -3\frac{3}{4}$

$\therefore y = \frac{3}{2}x - 3\frac{3}{4}$

d $CD = \sqrt{(4-2)^2 + (-1+2)^2} = \sqrt{(2)^2 + (1)^2} = \sqrt{5}$

e No, ABCD cannot be a rhombus as all 4 sides must be equal.

Question 3

a Let $\angle FAD = x$

$\Rightarrow \angle FDA = x$	(FA = FD)
$\angle EAB = x$	(vert opp angles)
$\angle EBA = x$	(EA = EB)
and let $\angle ABC = y$	
$\Rightarrow \angle ADC = y$	(opp angles parm)
$\therefore \angle EBC = x + y = \angle FDC$	

b $2x + 60° = 180°$

$x = 60°$

So both triangles AEB and FAD are equilateral triangles with

EB = AB = DC	(opp sides parm)
FD = AD = BC	(opp sides parm)
∠EBC = ∠FDC	(from *a*)
ΔEBC ≡ ΔCDF	(SAS)

Question 4

a i $\cos\theta = \frac{12}{13}$

ii $\tan\theta.\cos\theta = \frac{5}{13}$

iii $\theta = 22{,}62°$

b BC = 11 sin 43° = 7,5 cm

AB = 8,04 cm

Area = $\frac{1}{2}$.AB.BC

= 30,18 cm^2

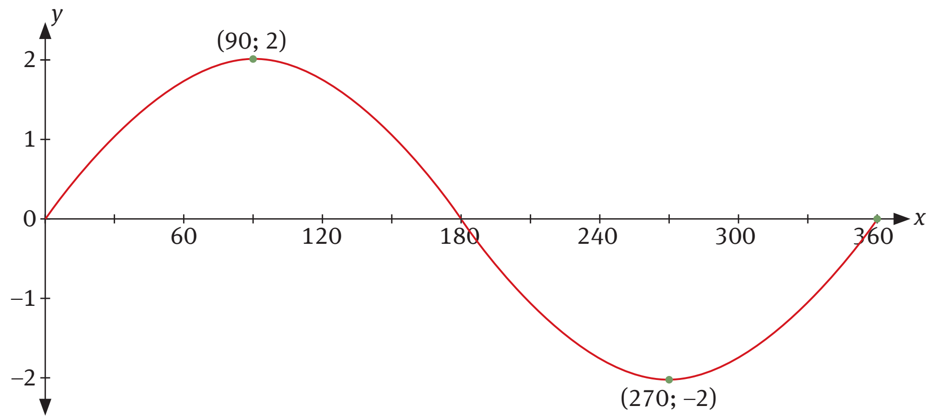
c i $\frac{BC}{40} = \tan 46°$

BC = 41,42 m

ii $\frac{41{,}42}{AC} = \tan 18°$

AC = 127,48 m

d



Question 5

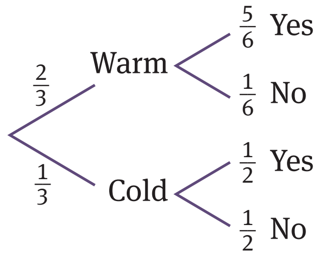
a i $\frac{13}{52} = \frac{1}{4}$

ii $\frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$

iii $\frac{4}{52} \times \frac{16}{51} + \frac{16}{52} \times \frac{4}{51}$

$= \frac{32}{663}$ or 0,0048

b



$\frac{2}{3} \times \frac{5}{6} = \frac{10}{18}$

$\frac{1}{3} \times \frac{1}{2} = \frac{3}{18}$

$\frac{10}{18} + \frac{3}{18} = \frac{13}{18}$

Question 6

a i $100 - 18 = 82$

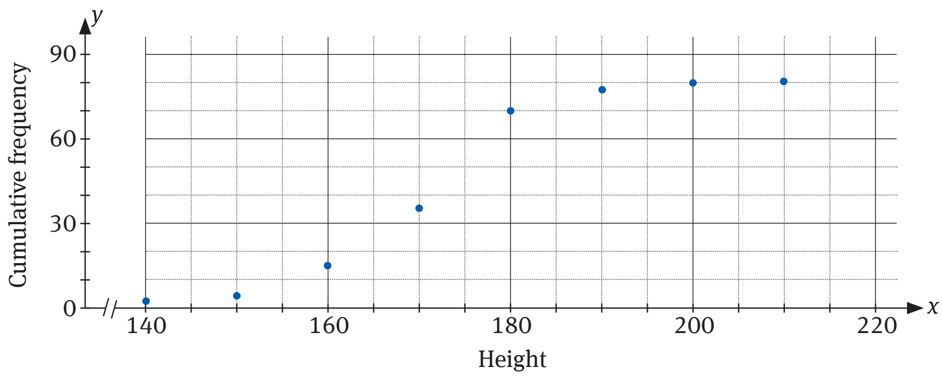
ii 41

iii $\frac{41 + 50}{2} = 45{,}5$

iv $\frac{1\,151}{22} = 52{,}32$

v $\frac{78 - 32}{2} = 23$

b i



ii Median ≈ 171 cm

Upper quartile ≈ 177 cm

iii 85% (80) = 68th value

≈ 180 cm

Glossary

A

adjacent Next to; in trigonometry, the adjacent side is the side next to a specified angle in a triangle

algebraic fractions An algebraic fraction is a fraction that has an algebraic expression in the numerator and the denominator

amplitude The amplitude of a graph is half the distance between the maximum and minimum points on the graph

angle of depression An angle of elevation is the angle between a horizontal line and the line from the observer to some object *below* the horizontal line

angle of elevation An angle of elevation is the angle between a horizontal line and the line from the observer to some object *above* the horizontal line

apex The highest point, or vertex

asymptote The points on the x-axis for which a function is undefined

B

base An **exponent** tells you how many times the **base** number is used as a factor

bearing A method of specifying direction using angles, starting from north at 0°; also called true bearing

binomial If a polynomial has two terms, we call it a binomial

C

Cartesian plane A plane divided by two axes on which all points are described by Cartesian coordinates (x and y)

circumference The circumference of the sphere is a circle around the middle of the sphere

class interval A group of data values that fall within a certain range; class intervals are useful when working with large numbers of data elements

clockwise In a circular direction, moving from right to left

coefficient If a number appears in front of a variable, then we call it the coefficient of that variable

collinear If three points are collinear, then they fall on the same straight line

common factors A common factor is one that divides into each term in an expression

complementary events Two sets are complementary when they are mutually exclusive and between them contain all the items in the sample space

compound interest With compound interest you earn interest on the original amount (the principal amount), plus any interest that you earn during the investment

congruent Two triangles are congruent if all their corresponding angles are the same size and their corresponding sides are equal in length

conjecture An opinion or conclusion formed on the basis of incomplete information

constant If a number appears on its own in an expression, then we call it a constant

continuous relationship A relationship in which all points on a line are possible solutions to the function or formula

conventional bearing Specifying a bearing using an angle and a compass point as a reference; also called direction

corresponding angles In similar triangles, the corresponding angles are the angles that are in the same relative positions and are the same size

corresponding sides In similar triangles, the corresponding sides are in the same relative positions

cos cosine; the trigonometric ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$

D

data Basic or raw values, without any meaning attached to the values

data elements A single data value in a data set

data points An individual data item, which will have a value

decreasing function In a decreasing function, the output variable *decreases* in value as the input variable *increases* in value

dependent variable In a function, the dependent variable depends on the value of the input variable, or independent variable

diameter The diameter is any line that passes through the centre of a circle or sphere from one side to the other

direction Use a directional compass (with points N, E, S and W) and an angle to specify direction; north is 0°; also referred to as conventional bearing

discrete relationship A relationship that results in a set of distinct points, rather than a continuous line

domain The domain of a function is the set of all x-values for which the function is defined

dual intercept method A method used to draw a straight line graph by joining the points that intercept the x- and y-axes

E

elements in a data set A data element is a single member of a data set

eliminate To remove

event a subset of the sample space

exchange rate Exchange rates give us a way of comparing different currencies so that countries can trade with each other

exhaustive A set of events is exhaustive if, between them, they contain all the possible events in the sample space

experimental probability The result of an experiment is the experimental probability or relative frequency

exponent An **exponent** tells you how many times the **base** number is used as a factor

exponential equations Equations in which the variables appear in the exponents of the terms

exponential function The graph of $y = e^x$ is upward-sloping, and increases faster as x increases; the graph always lies above the x-axis but can get arbitrarily close to it for negative x, which means that the x-axis is a horizontal asymptote

F

factorisation Factorisation is the reverse of the process of multiplying polynomials

first degree A linear equation is an equation of the first degree, which means that the highest power of the variable is 1

first difference the constant difference between the terms in a sequence is the first difference

frequency distribution A record of how many data elements fall within each class interval

Glossary

frequency table A table that shows each class interval and how many data elements fall within each interval

function Associates an input to a corresponding output according to a rule

G

general term We call the *n*th term the general term of the sequence

generalise To express an idea in general terms, as opposed to using specific examples

gradient A gradient is a slope

gradient intercept method A method used to draw a straight line graph using the slope (gradient) of the line

H

hemisphere A hemisphere is half of a sphere, cut through the middle

hire purchase A short-term loan agreement that lets you buy something now, but pay if off over a fixed number of months

hyperbola A hyperbola is a curve that has two pieces that are mirror images of each other and resemble two infinite bows

hypotenuse The longest side of a right-angled triangle

I

identity An equation that is true for every value of the variable is called an **identity** equation

increasing function In an increasing function, the output variable *increases* in value as the input variable *increases* in value

independent variable An input variable in a function that uniquely determines the value of the dependent variable

index This is another name for an exponent

inflation Inflation is the rate at which the price of goods increases annually

information When data is summarised or transformed in some way to give it meaning, it becomes information

input A formula takes in an input value, changes it according to a rule, and produces an output value

interquartile range The difference between the values of the first quartile and the third quartile; the range of the middle 50% of the data values

intersection the intersection of the two circles represents the event (A and B) or in set notation, $A \cap B$

inverse function We use the inverse functions on a calculator to find the value of an angle if we know the value of the trigonometric ratio ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$)

irrational number An irrational number is a number that we cannot
express as a fraction

isosceles trapezium An isosceles trapezium is a trapezium in which the non-parallel sides are equal in length

K

kite A quadrilateral with two pairs of adjacent sides equal in length

L

line segment The distance between two points is the length of the line segment between those two points

linear In a straight line

Glossary

linear equation A linear equation is an equation of the first degree, which means that the highest power of the variable is 1

linear inequality A linear inequality looks similar to an equation, but contains inequality signs rather than the equals sign: $\geq \leq < >$

linear number pattern A linear number pattern is a set or row of consecutive numbers in which the difference between the numbers remains the same irrespective of how many numbers there are in the pattern

literal equation An equation that contains more than one variable, meaning that we need to express one variable in terms of the others

lowest common denominator (LCD) the lowest number that all the denominators of a set of fractions can divide into

M

maximum The highest value in a data set

maximum value The highest *y*-value for which a function is defined

mean The result of dividing the sum of the data elements by the number of data elements; one of the measures of central tendency

measure of central tendency A value that reflects a typical or average member of a data set

measures of dispersion Summary values that show how spread out (dispersed) the data values are

median The data value that falls in the middle of an ordered set of data points; one of the measures of central tendency

midpoint The midpoint of a line segment between points A and B is the point half way along the line segment

minimum The lowest value in a data set

minimum value The lowest *y*-value for which a function is defined

mode The data value that appears most often in a data set; one of the measures of central tendency

monomial If a polynomial has a single term, we call it a monomial

mutually exclusive events two events are mutually exclusive if an outcome being part of one event means that it can never be part of the other

N

***n*th term** We call the *n*th term the general term of the sequence

O

opposite The side opposite a specified angle in a triangle

outcome The result of an experiment

output A formula takes in an input value, changes it according to a rule, and produces an output value

P

parabola A U-shaped graph formed by a quadratic function

parallel If two lines are parallel, then they have the same slope

parallelogram A quadrilateral with both pairs of sides parallel to each other

per annum Per year

percentile Percentiles divide a data set into 10 parts

period The period of a graph is the number of degrees in a complete cycle of the graph

Glossary

perpendicular Two lines are perpendicular if they intersect at 90°

polygon A figure enclosed by three or more line segments

polyhedron A polyhedron is a geometric solid in three dimensions with flat faces and straight edges

polynomial A polynomial is an algebraic expression that consists of two or more algebraic terms

power This is another name for an exponent

prime factors the prime factors of a number are all of the prime numbers that will exactly divide the given number

principal amount The amount originally invested or borrowed

probability Probability refers to the likelihood or chance of an event taking place

Q

quadrants A Cartesian plane is divided into four blocks by the two axes; each block is called a quadrant

quadratic A quadratic is a trinomial in the form $ax^2 + bx + c$

quadratic equation A quadratic equation is an equation of the second degree, which means that the highest power of the variable is 2

quadrilateral A quadrilateral is a two-dimensional figure with four sides

quartiles Quartiles divide a data set into four parts

R

radius The distance from the centre of a circle or sphere to the edge

range The range of a function is the set of all y-values for which the function is defined

range The difference between the highest and lowest data values

rational number A rational number is a number that we can write as a fraction, $\frac{a}{b}$, where a and $b \in$ Z, and $b \neq 0$

real number The set of real numbers includes all rational and irrational numbers

reciprocal ratio The reciprocal ratios are the inverse versions of each of the three main ratios, sin, cos and tan; they are cosec, sec and cot

rectangle A parallelogram with interior angles all equal to 90°

relative frequency The result of an experiment is the experimental probability or relative frequency

rhombus A parallelogram in which all sides are equal in length

right circular cone A right circular cone is a 3D shape formed by moving the end of a straight line in the shape of a circle while keeping the other end fixed above the centre of the circle

right prism A right prism is a polyhedron with two identical faces, called the bases

S

sample space The set of all possible outcomes

second degree A quadratic equation is an equation of the second degree, which means that the highest power of the variable is 2

semi-interquartile range The semi-interquartile range (S-IR) is the middle 50% of the interquartile range

Glossary

similar Two shapes are similar if their internal angles are the same, even if the shapes are different sizes

simple interest We calculate simple interest as a percentage of the original amount invested or borrowed only

simultaneous equation Solving two or more equations at the same time, which means finding values for the variables that satisfy all of the equations, not just one

sin sine; the trigonometric ratio $\frac{\text{oppsite}}{\text{hypotenuse}}$

slant height The length of the side of a cone

sphere A round 3D solid; the shape of a ball

square A rectangle in which all sides are equal in length

standard form The standard form of an equation is the form in which we usually write the equation – for example, the standard form of a linear equation is $y = mx + c$, and that of a quadratic is $ax^2 + bx + c$

substitution To replace one value or expression with another

surd A surd is a root of an integer that we cannot express as a fraction

surface area The surface area of a 3D solid refers to the area of the outside surface of the solid

symmetrical Two sets of points are symmetrical if they form a mirror image of each other about a line that runs between them

T

tan tangent; the trigonometric ratio $\frac{\text{opposite}}{\text{adjacent}}$

term We call each number in the sequence a term

trapezium A trapezium is a quadrilateral in which one pair of opposite sides is parallel

trigonometry Trigonometry is the field of Mathematics in which we study the relationship between the three sides and three angles of triangles

trinomial If a polynomial has two terms, we call it a trinomial

true bearing We measure the true bearing between two points in a clockwise direction starting from north; also referred to as bearing

turning point The point at which the slope of a graph is 0

U

union The set of elements in one event or in another event, (A or B), or in set notation, A B

V

variable A letter used in an expression that can take on different values

Venn diagram A Venn diagram is a graphical tool that we use to represent a sample space and sets of events

vertex A point on a polygon at which three or more sides meet

volume The volume of a solid refers to the amount of space inside the 3D solid